Some Conceptions Of Chaotic Properties Of Product Systems In Non-autonomous Discrete Systems

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ABSTRACT

This paper is devoted to a study of some chaotic properties of non-autonomous discrete dynamical systems, which are all defined in a way suitable for non-autonomous dynamical systems. It examines how properties such as mixing, transitive, weak mixing, mild mixing, strongly transitive, F −system, multi-transitive, or backward minimal, in a non-autonomous discretee dynamical system carry over to their product systems. However, it should be noted that while one of the notions above of the product system can be deduced from the factor systems.

1. Introduction

Dynamical systems theory is a useful mathematical tool for investigating a variety of natural and physical phenomena. The theory has been widely applied in several domains. As a result, many autonomous discrete dynamical system conclusions and approaches lack adequate approximations. In order to achieve better approximations, a system's governing rule must be time variant. Time−bvarying dynamical systems, or non-autonomous dynamical systems (NDS), have a major effect on the study of natural dynamics.

In 1996, Kolyada [1] first developed the notion of non-autonomous discrete dynamical systems (NDS). Let us denote by $\mathbb{N}$ for the positive integer numbers, and a compactttopological space $\mathcal{X}$ with a sequence $\hbar_{\mathrm{n}}: \mathcal{X} \to \mathcal{X}$ of continuous maps, such that $\mathrm{n} \in \mathbb{N}$. The pair $(\mathcal{X}, \hbar_{1,\infty})$ such that $\hbar_{1,\infty} = \{\hbar_{\mathrm{n}}\}_{\mathrm{n}=1}^{\infty}$ is a non-autonomous discrete dynamical systems (NDS). The orbit of the the point $z \in \mathcal{X}$ is $\mathrm{Orb}_{\hbar_{1,\infty}}(z) = \{z, \hbar_1^1(z), \hbar_1^2(z), \dots, \hbar_1^{\mathrm{n}}(z), \dots\}$, where $\hbar_1^{\mathrm{n}} = \hbar_{\mathrm{n}} \circ \hbar_{\mathrm{n}-1} \circ \dots \circ \hbar_2 \circ \hbar_1$.lLikewise, $\hbar_{\mathrm{r}}^{\mathrm{k}} = \hbar_{\mathrm{r}+\mathrm{k}-1} \circ \hbar_{\mathrm{r}+\mathrm{k}-2} \circ \dots \circ \hbar_{\mathrm{r}+1} \circ \hbar_{\mathrm{r}}$ for any $\mathrm{k}, \mathrm{r} \in \mathbb{N}$. Additionally, $\hbar_1^0 = \mathrm{id}_{\mathcal{X}}$ represent the identity mapping on $\mathcal{X}$. In recent years, numerous researchers have focused their research on the chaotic nature of (NDS).

In 2006, Tian [2] investigated devaney chaos in (NDS). In 2009, shi [3] introduced the concept of several types of chaotic properties in (NDS), such as transitivity, sensitivity, Li-Yorke chaos, etc. In 2011, C$\acute{a}$ novas [4] discussed the dynamic characteristics between Li-Yorke chaos and topological entropy in (NDS). In 2012, Balibree [5] examines the chaotic characteristics of (NDS). They used Bowen's method to define entropy for (NDS) and investigated the relationship between topological entropy and weak mixing. demonstrates that Li-Yorke chaos is not always implied by positive entropy. In 2017, authors [6] investigates the sensitivity properties of (NDS), providing counter examples and

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establishing conditions for various sensitivity types. In 2019, vasisht [7] discussed the relationships between various dynamical properties of (NDS) and their induced hyperspace systems, focusing on sensitivity, transitivity, and shadowing. In 2020, Salman [8] study topological and dynamical properties such as specification and chaos in (NDS) and their induced systems on hyperspaces and measure space. Also, study the notions of thick transitive and multi-transitive in (NDS) with minimal in [9]. In 2021, authors [10] study some kind of shadowing properties of the product system for hyperspace in (NDS). In 2022, authors [11] studied four kinds of shadowing propertis in (NDS). The collective sensitivity, collective accessibility, sensitivity, infinite sensitivity and Li–Yorke sensitivity on (NDS) are studied [12]. The shadowing property, transitivity, accessibility, and their relationships are studied in their product systems in (NDS) [13]. In 2025, Zong [15] study stronger forms of transitivity in (NDS). It challenges existing theorems, introduces new concepts, and provides counter examples to clarify the theory. In 2025, Chiranjeevi [16] explores varios notions of topological transitivitiy and their equivalences, implications, and relationships in (NDS) and generic dynamical system. In 2026, Al-Ameedi [17] discusses three types pf shadowing properties in general (NDS).

The authors in [6-8, 11-14, 17] studied the concept of product system for several chaotic properties like:(sensitive, transitive, point transitive, mixing, shadow, (periodic, strong, quasi-weak) specification properties, accessibility, etc). They show that if the system $(\mathcal{X}\times\mathcal{Y}, \hbar_{1,\infty}\times\mathcal{g}_{1,\infty})$ has one of the properties above then $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ also have the same property. So, our work in this paper aims to study and discuss the same concept of product system, but for other notions.

The structure of the paper is as follows: In the section two, will be present some preliminary findings and definition of some chaotic properties in (NDS). In.section three, the main results for the product system of (NDS) are studied.

2. Preliminaries

Let $\hbar_{1,\infty} = (\hbar_{\mathrm{n}})_{\mathrm{n}\in\mathbb{N}}$ and, $\mathcal{g}_{1,\infty} = (\mathcal{g}_{\mathrm{n}})_{\mathrm{n9}\in\mathbb{N}}$ be two sequences of self-maps on $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{X}, \mathcal{g}_{1,\infty})$ respectively. We will denoted by $(\hbar_{\mathrm{n}}\times\mathcal{g}_{\mathrm{n}})_{\mathrm{n}\in\mathbb{N}}$ as a sequence on $\mathcal{X}\times\mathcal{Y}$ of self-maps by $\hbar_{1,\infty}\times\mathcal{g}_{1,\infty}$ which is define as follow

$$(\hbar_1^{\mathrm{n}}\times\mathcal{g}_1^{\mathrm{n}})\big((\mathrm{x},\mathrm{y})\big) = (\hbar_{\mathrm{n}}\times\mathcal{g}_{\mathrm{n}})\circ(\hbar_{\mathrm{n}-1}\times\mathcal{g}_{\mathrm{n}-1})\circ\ldots\circ(\hbar_1.\times\mathcal{g}_1)(\mathrm{x},\mathrm{y}) \quad = (\hbar_1^{\mathrm{n}}(\mathrm{x}), \mathcal{g}_1^{\mathrm{n}}(\mathrm{y}))$$

for any $(\mathrm{x},\mathrm{y}) \in \mathcal{X}\times\mathcal{Y}$ and $\mathrm{n}\in\mathbb{N}$.

Remark2.1:. Let.$\mathcal{X},.\mathcal{Y}$ be topological space, let $\mathcal{A}_{\mathrm{i}}$, $\mathcal{B}_{\mathrm{i}}$ such that $\mathrm{i}\in\{1,2\}$ be non-empty subset of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Let $\mathrm{k}\in\mathbb{N}$, then

$$\big(\hbar_{1,\infty}\times\mathcal{g}_{1,\infty}\big)^{\mathrm{k}}(\mathcal{A}_{\mathrm{i}}\times\mathcal{B}_{\mathrm{i}}) = \hbar_{1,\infty}^{\mathrm{k}}(\mathcal{A}_{\mathrm{i}})\times\mathcal{g}_{1,\infty}^{\mathrm{k}}(\mathcal{B}_{\mathrm{i}}).$$

Definition2.2 [21]: For each $\mathrm{s}\in\mathbb{N}$, if $\hbar_1^{\mathrm{sn}}(z) = z$, then $z\in\mathcal{X}$ is called s $-$periodic point . If a point $z\in\mathcal{X}$ is $1-$ periodic, it is referred to as a fixed point.

Definition2. 3 [2, 18, 22]: For every open, nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{X}$. The system of NDS $(\mathcal{X}, \hbar_{1,\infty})$ is :

1- Transitive if there is n $\in \mathbb{N}$ suchtthat $\hbar_1^{\mathrm{n}}(\mathcal{A}) \cap \mathcal{B} \neq \emptyset$.

2- Mixing if there is $t \in \mathbb{N}$ and for any n $\geq t$, $\hbar_1^{\mathrm{n}}(\mathcal{A}) \cap \mathcal{B} \neq \emptyset$.

3- Strongly Transitive if there is s $\in \mathbb{N}$ such that $\bigcup_{\mathrm{i}=1}^{\mathrm{s}} \hbar_1^{\mathrm{i}}(\mathcal{A}) = \mathcal{X}$.

Definition 2. 4 [18]: A point z is point transitive if it has dense orbit in $\mathcal{X}$.

Definition 2. 5 [19]: Let $\mathcal{A}_1, \mathcal{A}_2$aand $\mathcal{B}_1, \mathcal{B}_2$ be opene, nonempty subsets in. $\mathcal{X}$..The system.of NDS $(\mathcal{X}, \hbar_{1,\infty})$ is saidtto be.WeaklyyMixing.if there exsits ni $\in \mathbb{N}$, thene

$$\hbar_1^{\mathrm{n}}(\mathcal{A}_1) \cap \mathcal{B}_1 \neq \emptyset \text{ and } \hbar_1^{\mathrm{n}}(\mathcal{A}_2) \cap \mathcal{B}_2 \neq \emptyset.$$

Definition2. 6: The system $(\mathcal{X}, \hbar_{1,\infty})$ is Backward Minimal if for every $t \in \mathcal{X}$, the subset $\{z \in \mathcal{X}: \hbar_1^{\mathrm{r}}(z) = t$ for some r $\in \mathbb{N}\}$ is denseiin $\mathcal{X}$.

Definition2. 7 [23]: The system $(\mathcal{X}, \hbar_{1,\infty})$ is totally transitive if, $\hbar_{1,\infty}^{[\mathrm{n}]}$is transitive for every n ≥ 1.

Definition2. 8: The system $(\mathcal{X}, \hbar_{1,\infty})$ is said to be F $-$system if per$(\hbar_{1,\infty})$ is dense in $\mathcal{X}$ and $\hbar_{1,\infty}$ is totally transitive.

Definition2. 9 [20]: Let $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ be transitive, if $(\mathcal{X}.\times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ isotransitive, then $(\mathcal{X}, \hbar_{1,\infty})$ is mildly mixing.

3. Main Results

Theorem 3. 1[**13**, theorem **3. 3**] **: Let** $(\boldsymbol{\mathcal{X}}, \boldsymbol{\hbar}_{\mathbf{1},\infty})$ **and** $(\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{g}}_{\mathbf{1},\infty})$ **be (NDS), if** $(\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}, \boldsymbol{\hbar}_{\mathbf{1},\infty} \times \boldsymbol{\mathcal{g}}_{\mathbf{1},\infty}$ **) is transitive point (resp. transitive, mixing) then** $(\boldsymbol{\mathcal{X}}, \boldsymbol{\hbar}_{\mathbf{1},\infty})$ **and** $(\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{g}}_{\mathbf{1},\infty})$ are both transitive point (resp. transitive, mixing).

Now we consider the following theorems for another notation for product system.

Theorem 3. 2: Let $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ be (NDS), if $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty}$) is weakly mixing, then $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ are both weaklypmixing

Proof: Suppose that $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ is weakly mixing. To prove that $(\mathcal{X}, \hbar_{1,\infty})$ is weakly mixing, we must show that $\hbar_{1,\infty} \times \hbar_{1,\infty}$ is transitive, let. $\mathcal{U}, \mathcal{V}$ be open, nonempty subsets of $\mathcal{X} \times \mathcal{X}$. Then, there exists an open, nonempty subsets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ of $\mathcal{X}$ with $\mathcal{A}_1 \times$ i$\mathcal{A}_2 \subseteq \mathcal{U}$, $\mathcal{B}_1 \times \mathcal{B}_2 \subseteq \mathcal{V}$. Let A, B $\subseteq \mathcal{Y}$ arelarbitrary open, nonempty subsets, then $(\mathcal{A}_1 \times$ A$) \times (\mathcal{A}_2 \times$ A) ands$(\mathcal{B}_1 \times$ B$) \times (\mathcal{B}_2 \times$ B) are open, nonempty subsets of $(\mathcal{X} \times \mathcal{Y}) \times (\mathcal{X} \times \mathcal{Y})$. Since the product system is weakly mixing, then there exists a positive integers n and a points

$$(a_1, b_1) \in \mathcal{A}_1 \times A, (a_2, b_2) \in t\mathcal{A}_2 \times A$$

such that

$(\hbar_1^n \times g_1^n)\,(a_1, b_1) \in \mathcal{B}_1 \times B$ and $(\hbar_1^n \times g_1^n)(a_2,\ b_2) \in \mathcal{B}_2 \times B$,

Therefore,

$$\hbar_1^n(a_1) \in \mathcal{B}_1\ , \hbar_1^n(a_2) \in \mathcal{B}_2.$$

Hence,

$$(\hbar_1^n \times \hbar_1^n)(a_1, a_2) \in \mathcal{B}_1 \times \mathcal{B}_2.$$

Since $(a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2 \subseteq \mathcal{U}$, we obtain

$$(\hbar_1^n \times \hbar_1^n\ )(\ \mathcal{U}) \cap \mathcal{V} \neq \emptyset,$$

which means that $\hbar_1 \times \hbar_1$ is transitive. Hence $(\mathcal{X}, \hbar_{1,\infty})$ is weakly mixing.

To prove that $(\mathcal{Y}, g_{1,\infty})$ is weakly mixing, let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ be open, nonempty subsets of $\mathcal{Y}$. Since $\mathcal{X}$ is nonempty, choose arbitrary open, nonoempty subsets $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathcal{X}$. Then, $\mathcal{W}_1 \times \mathcal{A}_1, \mathcal{W}_2 \times \mathcal{A}_2$are open, nonempty subsets of $\mathcal{X} \times \mathcal{Y}$. Similarly $\mathcal{W}_1 \times \mathcal{B}_1, \mathcal{W}_2 \times \mathcal{B}_2$are open, nonempty subsets of $\mathcal{X} \times \mathcal{Y}$. Since $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times g_{1,\infty})$ is weakly mixing, there exists a positive integers n such that

$$(\hbar_1 \times g_1)^n(\mathcal{W}_1 \times \mathcal{A}_1) \cap (\mathcal{W}_2 \times \mathcal{A}_2) \neq \emptyset.$$

and

$$(\hbar_1 \times g_1)^n(\mathcal{W}_1 \times \mathcal{B}_1) \cap (\mathcal{W}_2 \times \mathcal{B}_2) \neq \emptyset.$$

So, we have that the first relation becomes

$$(\hbar_1^n \times g_1^n)l(\mathcal{W}_1 \times \mathcal{A}_1) \cap (\mathcal{W}_2 \times \mathcal{A}_2) \neq \emptyset.$$

Therefore, there exists $x_1 \in \mathcal{W}_1$, $y_1 \in \mathcal{A}_1$such that

$$(\hbar_1^n(x_1), g_1^n(y_1)) \in \mathcal{W}_2 \times \mathcal{A}_2.$$

Hence, $\hbar_1^n(x_1) \in \mathcal{W}_2$and $g_1^n(y_1) \in \mathcal{A}_2$. In particular, $g_1^n(y_1) \in \mathcal{A}_2, y_1 \in \mathcal{A}_1$.thus,

$$g_1^n(\mathcal{A}_1\ ) \cap \mathcal{A}_2 \neq \emptyset.$$

Similarly, from

$$(\hbar_1^{\mathrm{n}} \times \mathcal{g}_1^{\mathrm{n}})\mathrm{l}(\mathcal{W}_1 \times \mathcal{B}_1) \cap (\mathcal{W}_2 \times \mathcal{B}_2) \neq \emptyset.$$

There exists $\mathrm{x}_2 \in \mathcal{W}_1$, $\mathrm{y}_2 \in \mathcal{B}_1$such that

$$(\hbar_1^{\mathrm{n}}(\mathrm{x}_2), \mathcal{g}_1^{\mathrm{n}}(\mathrm{y}_2)) \in \mathcal{W}_2 \times \mathcal{B}_2.$$

Consequently, $\mathcal{g}_1^{\mathrm{n}}(\mathrm{y}_2) \in \mathcal{B}_2$, $\mathrm{y}_2 \in \mathcal{B}_1$. Therefore

$$\mathcal{g}_1^{\mathrm{n}}(\mathcal{B}_1) \cap \mathcal{B}_2 \neq \emptyset.$$

Hence, for the same positive integers n,

$$\mathcal{g}_1^{\mathrm{n}}(\mathcal{A}_1) \cap \mathcal{A}_2 \neq \emptyset \text{ and } \mathcal{g}_1^{\mathrm{n}}(\mathcal{B}_1) \cap \mathcal{B}_2 \neq \emptyset.$$

since $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ were arbitrary open, nonemptycsubsets of $\mathcal{Y}$, we have that $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ is weakly mixing. Therefore $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ are both weakly mixing

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Theorem 3.3: Let $(\mathcal{X},, \hbar_{1,\infty})$ andf$(\mathcal{Y},, \mathcal{g}_{1,\infty})$ be (NDS), if $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ iss strongly transitive then $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, \mathrm{g}_{1,\infty})$ are both strongly transitive

Proof: Suppose that $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ is strongly transitive. Let $\mathcal{A} \subseteq \mathcal{X}$ be open, nonempty subsets. Since $\mathcal{Y}$ is open in itself, then $\mathcal{A} \times \mathcal{Y}$ is open, nonempty subsets of $\mathcal{X} \times \mathcal{Y}$. Since the system of the product is strongly transitive, then there exists $\mathrm{s} \in \mathbb{N}$ such that

$$\cup_{\mathrm{i}=1}^{\mathrm{s}}\left(\hbar_1^{\mathrm{i}} \times \mathcal{g}_1^{\mathrm{i}}\right)(\mathcal{A} \times \mathcal{Y}) = \mathcal{X} \times \mathcal{Y},$$

Since,

$$\left(\hbar_1^{\mathrm{i}} \times \mathcal{g}_1^{\mathrm{i}}\right)(\mathcal{A} \times \mathcal{Y}) = \hbar_1^{\mathrm{i}}(\mathcal{A}) \times \mathcal{g}_1^{\mathrm{i}}(\mathcal{Y}).$$

We obtain,

$$\cup_{\mathrm{i}=1}^{\mathrm{s}}(\hbar_1^{\mathrm{i}}(\mathcal{A}) \times \mathcal{g}_1^{\mathrm{i}}(\mathcal{Y})) = \mathcal{X} \times \mathcal{Y}.$$

Now, let $\mathrm{x} \in \mathcal{X}$ be arbitrary and for any $\mathrm{y} \in \mathcal{Y}$, then $(\mathrm{x}, \mathrm{y}) \in \mathcal{X} \times \mathcal{Y}$. Hence, there exists some $1 \leq \mathrm{i} \leq \mathrm{s}$. Such that $(\mathrm{x}, \mathrm{y}) \in \hbar_1^{\mathrm{i}}(\mathcal{A}) \times \mathcal{g}_1^{\mathrm{i}}(\mathcal{Y})$, therefore xo$\in \hbar_1^{\mathrm{i}}(\mathcal{A})$. Since x was arbitary,then $\cup_{\mathrm{i}=1}^{\mathrm{s}} \hbar_1^{\mathrm{i}}(\mathcal{A}) = \mathcal{X}$. Hence, $(\mathcal{X}, \hbar_{1,\infty})$ is strongly transitive.

Now, in the same way, for an open, nonempty subsets of $\mathcal{X} \times \mathrm{V}$, where $\mathcal{B} \subseteq \mathcal{Y}$, we obtain

$$\cup_{i=1}^{s} \mathcal{g}_1^{i}(\mathcal{B}) = \mathcal{Y}.$$

Hence, $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ is strongly transitive.

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Theorem 3.4: Let $(\mathcal{X}, \mathcal{h}_{1,\infty})$ andd$(\mathcal{Y}, \mathcal{g}_{1,\infty})$ebe (NDS), if $(\mathcal{X} \times \mathcal{Y}, \mathcal{h}_{1,\infty} \times \mathcal{g}_{1,\infty})$ is backward minimal then $(\mathcal{X}, \mathcal{h}_{1,\infty})$aand (Y, $\mathcal{g}_{1,\infty}$) are both backward minimal.

Proof: Suppose that $(\mathcal{X} \times \mathcal{Y}, \mathcal{h}_{1,\infty} \times \mathcal{g}_{1,\infty})$ is backward minimal. To prove that $(\mathcal{X}, \mathcal{h}_{1,\infty})$ is backward minimal. Let, $\mathrm{x}_0 \in \mathcal{X}$ and let $\mathcal{A}$ be open, nonempty subsets of $\mathcal{X}$, for any arbitrary point $\mathrm{y}_0 \in \mathcal{Y}$, hence $\mathcal{A} \times \mathcal{Y}$ be open, nonempty subsets of $\mathcal{X} \times \mathcal{Y}$. By hypothesis, we deduce that

$$\{(\mathrm{x}, \mathrm{y}) \in \mathcal{X} \times \mathcal{Y}: (\mathcal{h}_1 \times \mathcal{g}_1)^{\mathrm{r}}(\mathrm{x}, \mathrm{y}) = (\mathrm{x}_0, \mathrm{y}_0), \text{for some } \mathrm{r} \in \mathbb{N}\}.$$

is dense in $\mathcal{X} \times \mathcal{Y}$. Hence, therekexists $(\mathrm{u}, \mathrm{v}) \in \mathcal{A} \times \mathcal{Y}$ and $\mathrm{r} \in \mathbb{N}$ suchathat

$$(\mathcal{h}_1 \times \mathcal{g}_1)^{\mathrm{r}}(\mathrm{u}, \mathrm{v}) = (\mathrm{x}_0, \mathrm{y}_0).$$

It follows that

$$\mathcal{h}_1^{\mathrm{r}}(\mathrm{u}) = \mathrm{x}_0, \mathcal{g}_1^{\mathrm{r}}(\mathrm{v}) = \mathrm{y}_0.$$

Since $\mathrm{u} \in \mathcal{A}$, thuss

$$\mathrm{u} \in \{\mathrm{x} \in \mathcal{X}: \mathcal{h}_1^{\mathrm{r}}(\mathrm{x}) = \mathrm{x}_0, \text{for some } \mathrm{r} \in \mathbb{N}\} \cap \mathcal{A}.$$

Hence ,

$$\{\mathrm{x} \in \mathcal{X}: \mathcal{h}_1^{\mathrm{r}}(\mathrm{x}) = \mathrm{x}_0, \text{for some } \mathrm{r} \in \mathbb{N}\} \cap \mathcal{A} \neq \emptyset.$$

Thus, the set

$$\{\mathrm{x} \in \mathcal{X}: \mathcal{h}_1^{\mathrm{r}}(\mathrm{x}) = \mathrm{x}_0, \text{for some } \mathrm{r} \in \mathbb{N}\}$$

is dense in $\mathcal{X}$. therefore $(\mathcal{X}, \mathcal{h}_{1,\infty})$ is backward minimal.

Similarly, Let, $\mathrm{y}_0 \in \mathcal{Y}$ and let $\mathcal{B}$ be open, nonempty subsets of $\mathcal{Y}$, choose a point $\mathrm{x}_0 \in \mathcal{X}$. Then $\mathcal{X} \times \mathcal{B}$ is open, nonempty subsets of $\mathcal{X} \times \mathcal{Y}$. Again, By backward minimality of the system, there exist $(\mathrm{u}, \mathrm{v})\mathrm{u} \in \mathcal{X} \times \mathcal{B}$ and $\mathrm{r} \in \mathbb{N}$ suchhthat

$$(\hbar_1 \times \mathcal{g}_1)^{\mathrm{r}}\,(\mathrm{u},\mathrm{v}) = (\mathrm{x}_0, \mathrm{y}_0),$$

consequently, $\mathcal{g}_1^{\mathrm{r}}(\mathrm{v}) = \mathrm{y}_0$, with $\mathrm{v} \in \mathcal{B}$, it follows that

$$\{\,\mathrm{y} \in \mathcal{Y}: \mathcal{g}_1^{\mathrm{r}}(\mathrm{y}) = \mathrm{y}_0 \text{ for some } \mathrm{r} \in \mathbb{N}\} \cap \mathcal{B} \neq \emptyset.$$

Hence, since $\mathcal{B}$ is arbitrary. Thus, the set

$$\{\,\mathrm{y} \in \mathcal{Y}: \mathcal{g}_1^{\mathrm{r}}(\mathrm{y}) = \mathrm{y}_0 \text{ for some } \mathrm{r} \in \mathbb{N}\}$$

is dense in $\mathcal{Y}$. therefore $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ is backward minimal.

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Theorem3. 5: Let $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ be (NDS), if $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ is totally transitive, then $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ are both totally transitivee

Proof : uppose that $\hbar_{1,\infty} \times \mathcal{g}_{1,\infty}$ is totally transitive, then $(\hbar_{1,\infty} \times \mathcal{g}_{1,\infty})^{\mathrm{s}}$ is transitive, which means $\hbar_{1,\infty}^{\mathrm{s}} \times \mathcal{g}_{1,\infty}^{\mathrm{s}}$ is transitive, by theorem 3.1 we have that $\hbar_{1,\infty}^{\mathrm{s}}$ and $\mathcal{g}_{1,\infty}^{\mathrm{s}}$are both transitive. Therefore, $\hbar_{1,\infty}$and $\mathcal{g}_{1,\infty}$ are totally transitive.

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Theorem3. 6: Let $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ be (NDS), thehset of periodicopoints of $\hbar_{1,\infty} \times \mathcal{g}_{1,\infty}$ is dense in $\mathcal{X} \times \mathcal{Y}$ if and only if the set of periodic points of $\hbar_{1,\infty}$ is dense in $\mathcal{X}$and the set of periodic points of $\mathcal{g}_{1,\infty}$ is dense in $\mathcal{Y}$

Proof : Let A, B be open, nonempty subsets of,$\mathcal{X}$ and $\mathcal{Y}$ respectively, then $\mathrm{A} \times \mathrm{B}$ is open, nonempty subsets of $\mathcal{X} \times \mathcal{Y}$. Since the set of periodic points in $\hbar_{1,\infty} \times \mathcal{g}_{1,\infty}$ is dense in $\mathcal{X} \times \mathcal{Y}$. Then there exists $(\mathcal{p}, \mathcal{q}) \in \mathrm{A} \times \mathrm{B}$ and$\mathrm{r} \in \mathbb{N}$, such that

$$(\hbar_1 \times \mathcal{g}_1)^{\mathrm{kr}}(\mathcal{p}, \mathcal{q}) = \left(\hbar_1^{\mathrm{kr}}(\mathcal{p}), \mathcal{g}_1^{\mathrm{kr}}(\mathcal{q})\right) = (\mathcal{p}, \mathcal{q}) \text{for any } \mathrm{k} \in \mathbb{N}.$$

Hence, we obtain that

$$\hbar_1^{\mathrm{kr}}(\mathcal{p}) = \mathcal{p} \text{ for } \mathcal{p} \in \mathrm{A} \text{ and } \mathcal{g}_1^{\mathrm{kr}}(\mathcal{q}) = \mathcal{q} \text{ for } \mathcal{q} \in \mathrm{B}.$$

So, it means that the set of periodic points of,$\hbar_{1,\infty}$ is dense in $\mathcal{X}$and the set of periodic pointshof $\mathcal{g}_{1,\infty}$ is dense in $\mathcal{Y}$.

Assume that $\hbar_{1,\infty}$ and $\mathcal{g}_{1,\infty}$ have dense periodic point in $\mathcal{X}$ and $\mathcal{Y}$ respectively, and let $\mathcal{U} \subset \mathcal{X} \times \mathcal{Y}$ be open, nonempty subsets. Then, there exists a nonempty open sets $\mathrm{A} \subset \mathcal{X}$ and $\mathrm{B} \subset \mathcal{Y}$ such that $\mathrm{A} \times \mathrm{B} \subset \mathcal{U}$. By hypothesis there exests $\mathcal{p} \in \mathrm{A}$ such that $\hbar_1^{\mathrm{kr}}(\mathcal{p}) = \mathcal{p}, \mathrm{r} \in \mathbb{N}$ and there exests $\mathcal{q} \in \mathrm{B}$ such that $\mathcal{g}_1^{\mathrm{lr}}(\mathcal{q}) = \mathcal{q}, \mathrm{r} \in \mathbb{N}$. Let, $(\mathcal{p}, \mathcal{q}) \in \mathcal{U}$, we let $\mathrm{m} = \mathrm{kl}$, we obtain that

$$(\hbar_1 \times \mathcal{g}_1)^{\mathrm{m}}(\mathcal{p}, \mathcal{q}) = \left(\hbar_1^{\mathrm{m}}(\mathcal{p}), \mathcal{g}_1^{\mathrm{m}}(\mathcal{q}) \right) = (\mathcal{p}, \mathcal{q}).$$

Hence, $\mathcal{U}$ contain periodic point andithe set of periodic points of $\hbar_1 \times \mathcal{g}_1$ isudense in $\mathcal{X} \times \mathcal{Y}$.

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Theorem3.7: Let $(\mathcal{X}, \hbar_{1,\infty})$ andd$(\mathcal{Y}, \mathcal{g}_{1,\infty})$ be (NDS), if $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ is satisfy the $\mathrm{F}-$ system property, then $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ are both an $\mathrm{F}-$ system .

Proof: Suppose that $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ is satesfy the F- system property. Thus, $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ is totally transitive and $\mathrm{per}(\hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ is dense in $\mathcal{X} \times \mathcal{Y}$. By theorem(3.5), we have $\hbar_{1,\infty}$and $\mathcal{g}_{1,\infty}$ is totally transitive, moreover, by theorem (3.6) $\mathrm{per}(\hbar_{1,\infty})$ is dense in $\mathcal{X}$ and $\mathrm{per}(\mathcal{g}_{1,\infty})$ is dense in $\mathcal{Y}$. Therefore, we have $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ are both an $\mathrm{F}-\mathrm{system}$.

■

Theorem 3.8: Let $(\mathcal{X}, \hbar_{1,\infty})$ andg$(\mathcal{Y}, \mathcal{g}_{1,\infty})$ be (NDS), then $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ is mild mixing if and only if $\left(\mathcal{X}, \hbar_{1,\infty}\right)$ and $(\mathcal{Y}, \mathcal{g}_{1,\infty})$ are both mild mixing.

Proof : Suppose that $(\mathcal{X} \times \mathcal{Y}, \hbar_{1,\infty} \times \mathcal{g}_{1,\infty})$ is mild mixing. By Defintion (2.9), let $(\mathrm{Z}, \mathrm{h}_{1,\infty})$ be transitive (NDS) and $\mathrm{h}_{1,\infty}: \mathrm{Z} \to \mathrm{Z}$, such that $((\mathcal{X} \times \mathcal{Y}) \times \mathrm{Z}, (\hbar_{1,\infty} \times \mathcal{g}_{1,\infty}) \times \mathrm{h}_{1,\infty})$ is transitive. To prove that $(\mathcal{X} \times \mathrm{Z}, \hbar_{1,\infty} \times \mathrm{h}_{1,\infty})$ is transitive. Let $\mathcal{U}$ and $\mathcal{V}$ be open, nonempty susets of $\mathcal{X} \times \mathrm{Z}$. Then, there exists an open, nonempty subsets $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{X}$, and $\mathrm{W}_1, \mathrm{W}_2$ of Z such that

$$\mathcal{A}_1 \times \mathrm{W}_1 \subseteq \mathcal{U} \text{ and } \mathcal{A}_2 \times \mathrm{W}_2 \subseteq \mathcal{V}.$$

Then, for any open, nonempty subset $\mathcal{B} \subseteq \mathcal{Y}$, hence

$$(\mathcal{A}_1 \times \mathcal{B}) \times \mathrm{W}_1 \text{ and } (\mathcal{A}_2 \times \mathcal{B}) \times \mathrm{W}_2$$

are open, non-empty of $(\mathcal{X} \times \mathcal{Y}) \times \mathrm{Z}$. Since $((\mathcal{X} \times \mathcal{Y}) \times \mathrm{Z}, (\hbar_1 \times \mathcal{g}_1) \times \mathrm{h}_1)$ is transitive, there exists

$$((\mathrm{u}, \mathrm{v}), \mathrm{w}) \in (\mathcal{A}_1 \times \mathcal{B}) \times \mathrm{W}_1 \text{ and } \mathrm{n} \in \mathbb{N}$$

such that

$$((\hbar_1 \times \mathcal{g}_1) \times \mathrm{h}_1)^{\mathrm{n}}((\mathrm{u}, \mathrm{v}), \mathrm{w}) \in (\mathcal{A}_2 \times \mathcal{B}) \times \mathrm{W}_2.$$

We obtain $\hbar_1^n(u) \in \mathcal{A}_2$ and $h_1^n(w) \in W_2$. Therefore,

$$(\hbar_1 \times h_1)^n(u, w) \in \mathcal{A}_2 \times W_2 .$$

since $(u, w) \in \mathcal{A}_1 \times W_1$, we have that

$$(\hbar_1 \times h_1)^n(\mathcal{A}_1 \times W_1) \cap (\mathcal{A}_2 \times W_2) \neq \emptyset.$$

Consequently,

$$(\hbar_1 \times h_1)^n(\mathcal{U}) \cap \mathcal{V} \neq \emptyset.$$

Thus, $(\mathcal{X} \times Z, \hbar_{1,\infty} \times h_{1,\infty})$ is transitive. It follows that $(\mathcal{X}, \hbar_{1,\infty})$is mildly mixing.

Similarly, we can that obtain that $(\mathcal{Y}, g_{1,\infty})$is mildly mixing by interchanging the roles of $\mathcal{X}$ and $\mathcal{Y}$. Hence $(\mathcal{X}, \hbar_{1,\infty})$ and $(\mathcal{Y}, g_{1,\infty})$ are mildly mixing.

■

4. Conclusions

The concept of a product system in a non-autonomous dynamical system is presented in this paper. We find that if the system $(\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}, \boldsymbol{\hbar_{1,\infty}} \times \boldsymbol{g_{1,\infty}})$ is one of the following classes of functions: weak mixing, strong transitive, mild mixing, F-system, totally transitive, or backward minimal. then, each of $(\boldsymbol{\mathcal{X}}, \boldsymbol{\hbar_{1,\infty}})$ and $(\boldsymbol{\mathcal{Y}}, \boldsymbol{g_{1,\infty}})$ have the same property of the above notations.

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