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Coefficient Estimates for Certain New Subclasses of Bi-Univalent Functions Associated with Quasi-Subordination

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ABSTRACT

An investigation into a novel subclass of analytic and bi-univalent functions associated with quasi-subordination in the open unit disk is introduced. The factor estimates $|a_2|, |a_3|$, for functions in belonging to these subclasses are established. Furthermore, several relevant consequences and special cases of the main results are highlighted.

1. Introduction

Univalent function theory is a key area of complex analysis. Researchers study how analytic functions map the open unit disk. In this domain, bi-univalent functions play a central role. Both the function and its inverse map must be univalent. This requirement creates strong geometric properties. Over the years, scholars have explored many special subclasses. Our work builds directly on these foundational concepts. Let A denote the class of functions that are analytic in the open unit disk of the from:

$$f(v) = v + \sum_{j=2}^{\infty} a_j v^j, \quad (v \in \mathfrak{U}) \tag{1.1}$$

Which are A. the open unit disc, $\mathfrak{U} = \{v \in \mathbb{C} : |v| < 1\}$. Additionally, consider the subclass comprising functions that are univalent throughout the open unit disk. Standard univalence theory guarantees that every such function possesses a well-defined inverse map defined over a region encompassing at least the quarter-disk centered at the origin. This inverse mapping can subsequently be expressed through its Maclaurin power series expansion:$f^{-1}(f(v)) = v \quad , (v \in \mathfrak{U})$,
and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \tfrac{1}{4}). \tag{1.2}$$

Actually, the inverse function, $g = f^{-1}$ can be written by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.3}$$
$$= w + \sum_{n=2}^{\infty} A_n w^n$$

The function $f \in \mathcal{A}$ is termed bi-u in $\mathfrak{U}$ if both f and its inverse f^{-1} are u functions in $\mathfrak{U}$ given by (1.1). The study of bi-univalent functions originated with Lewin [5], who established that $|a_2| \leq 1.51$ for functions satisfying (1.1). Subsequently, Brannan and Clunie [2] posed the conjecture that $|a_2| \leq \sqrt{2}$. While Netanyahu [7] later demonstrated that $\max_{f \in \sum} |a_2| = \frac{4}{3}$. Early

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studies on bi-univalent functions focused on coefficient bounds. Lewin introduced this class and established the first upper bound. Soon after, Brannan and Clunie proposed a famous conjecture. Netanyahu later resolved this problem by finding the exact maximum value. Today, researchers extend these classical results using modern tools. We combine these ideas with quasi-subordination to create broader function classes. Also different authors have studied classes of bi $u\ A.f.$ and obtained approximation of the factors $|a_2|$ and $|a_3|$, for $f.$ in these classes. For two $A.f.$ f and $g,$ f is subject to g, take the following form:

$f(v) \prec g(v)\ (v \in \mathfrak{U})$.,… (1.4)

if $A.f.$ exist hand h and $\hbar(v)$, using$|\hbar(z)| \leq 1$

$$\hbar(0) = 0\ and\ |\hbar(v)| < 1(v \in \mathfrak{U}),$$

For which

$$f(v) = h(v)g\big(\hbar(v)\big), (v \in \mathfrak{U}).$$

Observe that if($h(\mathrm{v}) = 1$), thus $f(\mathrm{v}) = g\big(\hbar(\mathrm{v})\big)$, hance $f(\mathrm{v}) \prec g(v)$ $(z \in \mathfrak{U})$.I u in $\mathfrak{U}$, $f \prec g$ $\leftrightarrow$ $f(0) = g(0)$ and $f(\mathfrak{U}) \subset g(\mathfrak{U})$.

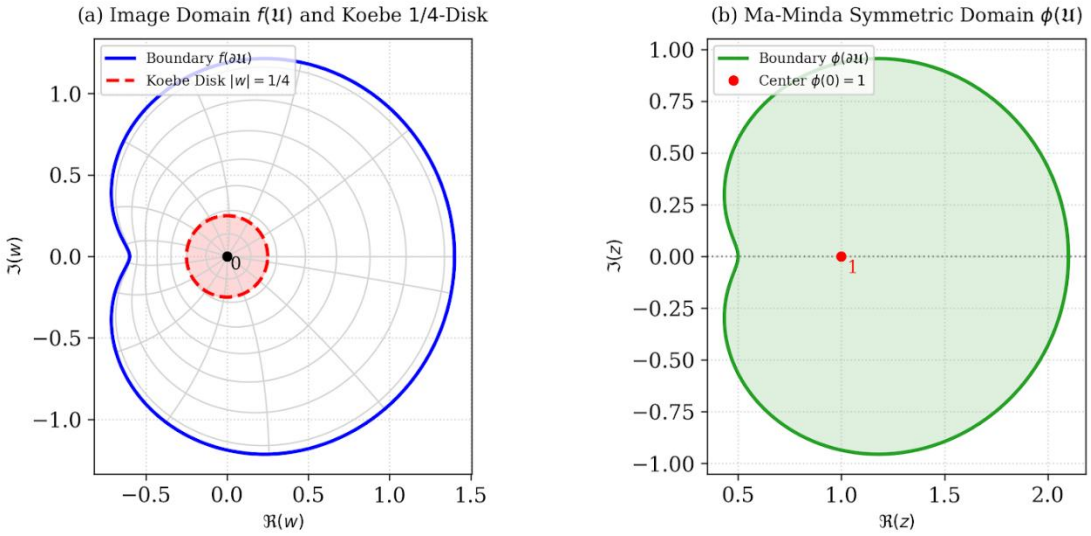


Figure 1: Geometric representations of the underlying domains: (a) Image domain $f(\mathfrak{U})$ of a univalent function containing the guaranteed Koebe disk $|\mathrm{w}| < \frac{1}{4}$ where f^{-1} is defined; (b) Typical Ma-Minda target domain $\phi(\mathfrak{U})$ exhibiting starlikeness with respect to $\phi(0) = 1$ and symmetry about the real line.

Ma and Minda introduced a combined treatment of different subclass composed of starlike with convex $f.$ for one of quantities $\frac{vf'(v)}{f(v)}$ or $1 + \frac{vf''(v)}{f(v)}$ is continued to a broader scope superordinate $f.$ The $S^*(\phi)$ presented by Ma and Minda [5]. Composed of function $f \in \mathcal{A}$ fulfilling $\frac{vf'(v)}{f(v)} \prec \theta(v), v \in \mathfrak{U}$ moreover corresponding class $k(\theta)$ of convex functions $f \in \mathcal{A}$ attaining $1 + \frac{vf''(v)}{f(v)} \prec \phi(v), v \in \mathfrak{U}$, Ma and Minda [6]. Interestingly this class has passed through several developments. These stages include exploration into bi-univalent functions by bounding the second coefficient and stage of a smaller upper bound for this parameter. Then, came the the

conjecture, proving the precise peak value across the entire class. The most important of these stages is the study into starlike and convex collections together by forcing key differential quotients to lie inside a univalent region, as seen in Figure 1. Also, Figure 2 gives a good picture into the intersection point between two separate expressions.

$$\phi(v) = 1 + B_1 v + B_2 v^2 + B_3 v^3 + \cdots, \quad B_1 > 0. \tag{1.5}$$

with

$$h(v) = h_0 + hv + h_2 v^2 + h_3 v^3 + \cdots. \tag{1.6}$$

We focus on this because it can help us later introduce important subfamilies and derive their explicit upper bounds for their initial Taylor–Maclaurin coefficients. Furthermore, from this we can thoroughly explore the relevant consequences and special cases.

Lemma (1.1)[8]: Let $p(v) \in \mathcal{P}$,then $|pi| \leq 2$ for each i where $\mathcal{P}$ is the family of all functions p, analytic in $\mathfrak{U}$, for which $Re(p(v)) > 0, (v \in \mathfrak{U})$, where

$$p(v) = 1 + p_1 v + p_2 v^2 + p_3 v^3 + \cdots,$$

2. Initial Coefficient Inequalities for the General Bi-Univalent Subclass

This subclass gives us a handy tool to combine different function families by joining basic subordination with a bounded analytic factor. We will try to shift some constants around to diverse our subclasses and their conditions. As we will see later, this helps find starlike and convex properties have tighter coefficient bounds.

Definition $(\mathbf{2.1})$: The subclass $\not{\mathcal{F}}_{q,\Sigma}^{\nu}(\delta, \eta, \theta)$ consists of all analytic bi-univalent functions \$f\$ governed by the dual conditions:

$$\left[\frac{v[f'(v) + \delta v f''(v)]^{\nu}}{\delta v(f'(v) + \eta f''(v)) + (1-\delta)((1-\eta)f(v) + \eta v f'(v)}\right] - 1 \prec_q (\theta(v) - 1) \tag{2.1}$$

with

$$\left[\frac{w[g'(w) + \delta w g''(w)]^{\nu}}{\delta w(g'(w) + \eta g''(w)) + (1-\delta)((1-\eta)g(w) + \eta w g'(w)}\right] - 1 \prec_q (\theta(\mathrm{w}) - 1) \tag{2.2}$$

Where $0 \leq \delta \leq 1, 0 \leq \nu \leq 1, 0 \leq \delta \leq 1$ and $\mathrm{g} = f^{-1}$is given by (1.3) and $(v, w \in \mathfrak{U})$.

Hence, we notice that for special values to our parameters ν, η, δ and θ can give us new classes. These classes simplify the messy conditions and help us give better conditions and relationships to other classes.

Remark (2.1): By specializing the parameters in Definition 2.1, several noteworthy subclasses emerge:

1- For $\eta = 0$, a function $f \in \sum$ define by (1.1) is regarded in the class $\not{\mathcal{F}}_{q,\Sigma}^{\nu}(\delta, \eta, \theta)$ if the subsequent conditions are fulfilling:

$$\left[\frac{v[f'(v) + \delta v f''(v)]^{\nu}}{\delta v f'(v) + +(1-\delta)f(v)}\right] - 1 \prec_q (\theta(v) - 1),$$

and

$$\left[\frac{w[g'(w) + \delta w g''(w)]^{\nu}}{\delta w g'(w) + (1-\delta)g(w)}\right] - 1 \prec_q (\theta(\mathrm{w}) - 1),$$

2- For $\eta = 1$, a function $f \in \sum$ define by (1.1) is called to be in the class $\not{\mathcal{F}}_{q,\Sigma}^{\nu}(\delta, \eta, \theta)$ if the subsequent conditions are fulfilling:

$$\left[\frac{v[f'(v)+\delta vf''(v)]^{\nu}}{\delta vf''(v)+(1-\delta)vf'(v)}\right]-1 \prec_q (\theta(v)-1),$$

and

$$\left[\frac{w[g'(w)+\delta wg''(w)]^{\nu}}{\delta wg''(w)+(1-\delta)wg'(w)}\right]-1 \prec_q (\theta(\mathrm{w})-1),$$

Where $0 \leq \delta \leq 1, 0 \leq \nu \leq 1, 0 \leq \delta \leq 1$ and g = $f^{-1}$is determined by (1.3) and $(v, w \in \mathfrak{U})$.

3-For $\eta = 0$ and $\nu = 1$,we get the class $\mathcal{F}_{q,\Sigma}^{1}(\delta, 0, \theta) := S_{\Sigma}^{*,q}(\phi)$**,**which was considered by [2]

4-For $\eta = 0, \delta = 1$, we get the class $\mathcal{F}_{q,\Sigma}^{1}(1,0,\theta) := S_{\Sigma}^{*,q}(\phi)$**,**which was considered by [2]

Subsequent, we obtain that estimates for the factors $|a_2|$ and $|a_3|$ for the f.in $\mathcal{F}_{q,\Sigma}^{\nu}(\delta, \eta, \theta)$. From this result we can present a theorem with upper bounds for the second and third coefficients which depend directly on the chosen parameters. This can be done by noticing that the third coefficient bound involves a minimum of two separate expressions.

Theorem $(\mathbf{2.1})$: Let f be a function in the subclass $\mathcal{F}_{q,\Sigma}^{\nu}(\delta, \eta, \theta)$. Then the initial Taylor–Maclaurin coefficients satisfy the bounds

$$|a_2| \leq min\left\{\frac{|A_0|B_1}{(1+\delta)(2\nu-\eta-1)}, \frac{\sqrt{|A_0|\,(B_1+|B_2-B_1|}}{\sqrt{[(1+\delta)^2[2\nu(\nu-1)-(2\nu-\eta-1)(1+\eta)]+2(1+2\delta)(3\nu-2\eta-1)]}}\right\},$$

(2.3)

and

$$|a_3| \leq min\left\{\frac{B_1(|A_0|+|A_1|)}{(1+2\delta)(3\nu-2\eta-1)}+\frac{A_0^2B_1^2}{(1+\delta)^2(2\nu-\eta-1)^2}, \frac{|A_0|\,(B_1+|B_2-B_1|)}{[(1+\delta)^2[2\nu(\nu-1)-(2\nu-\eta-1)(1+\eta)]+2(1+2\delta)(3\nu-2\eta-1)]}+\frac{B_1(|A_0|+|A_1|)}{(1+2\delta)(3\nu-2\eta-1)}\right\}.$$

(2.4)

Proof : Let $f \in \mathcal{F}_{q,\Sigma}^{\nu}(\delta, \eta, \theta)$ and $g = f^{-1}$ be defined as in (1.3). By the definition of quasi-subordination, there exist analytic Schwarz functions $u, v : \mathfrak{U} \longrightarrow \mathfrak{U}$ satisfying $u(0) = v(0) = 0, |u(v)| < 1, |v(w)| < 1$ and a function h given by (1.5), such that

$$\left[\frac{v[f'(v)+\delta vf''(v)]^{\nu}}{\delta v(f'(v)+\eta f''(v))+(1-\delta)((1-\eta)f(v)+\eta vf'(v)}\right]-1 \prec_q h(v)(\theta(u(v)-1)), v \in u \quad (2.5)$$

and

$$\left[\frac{w[g'(w)+\delta wg''(w)]^{\nu}}{\delta w(g'(w)+\eta g''(w))+(1-\delta)((1-\eta)g(w)+\eta wg'(w)}\right]-1 \prec_q h(w)(\theta(v(w)-1)), w \in u. (2.6)$$

We now introduce the helper functions $\boldsymbol{m}(\boldsymbol{z})$ and $\boldsymbol{n}(\boldsymbol{w})$, expressed as

$$m(v) = \frac{1+u(v)}{1-u(v)} = 1 + c_1 v^2 + c_2 v^2 + \cdots \quad (2.7)$$

and

$$n(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1 w^2 + d_2 w^2 + \cdots. \quad (2.8)$$

Or in other words

$$u(v) := \frac{m(v)-1}{m(v)+1} = \frac{1}{2}[c_1 v + \left(c_2 - \frac{c_1^2}{2}\right) v^2 + \cdots \quad (2.9)$$

and

$$v(w) := \frac{n(w)-1}{n(w)+1} = \frac{1}{2}[b_1 w + \left(b_2 - \frac{b_1^2}{2}\right) w^2 + \cdots. \quad (2.10)$$

Next, we equate terms with identical powers. This step creates a system of algebraic equations involving the initial coefficients. By subtracting one relation from another, we isolate the second coefficient. We then use the magnitude restrictions on the Schwarz functions. Simple arithmetic leads directly to our main inequality. The same approach yields the bound for the third coefficient. Using (2.9),(2.10) in (2.5) (2.6),respectively, we have

$$\left[\frac{v[f'(v)+\delta v f''(v)]^{\nu}}{\delta v(f'(v)+\eta f''(v))+(1-\delta)((1-\eta)f(v)+\eta z f'(v)}\right] - 1$$
$$= h(v)\left(\phi\left(\frac{p(v)-1}{p(v)+1}\right)-1\right), \quad (2.11)$$

and

$$\left[\frac{w[g'(w)+\delta w g''(w)]^{\nu}}{\delta w(g'(w)+\eta g''(w))+(1-\delta)((1-\eta)g(w)+\eta w g'(w)}\right] - 1 = h(w)\left(\phi\left(\frac{q(w)-1}{q(w)+1}\right)-1\right). \quad (2.12)$$

Substituting series expansions (1.5) and (1.6) into the right-hand sides of equations (2.11) and (2.12), we obtain

$$h(v)\left(\phi\left(\frac{p(v)-1}{q(v)+1}\right)-1\right) = \frac{1}{2}A_0 B_1 c_1 v + \left\{\frac{1}{2}A_1 B_1 c_1 + \frac{1}{2}A_0 B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0 B_2}{4}c_1^2\right\} v^2 + \cdots. \quad (2.13)$$

and

$$h(w)\phi\left(\left(\frac{q(w)-1}{q(w)+1}\right)-1\right)$$
$$= \frac{1}{2}A_0 B_1 d_1 w + \left\{\frac{1}{2}A_1 B_1 d_1 + \frac{1}{2}A_0 B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{A_0 B_2}{4}d_1^2\right\} w^2 + \cdots. \quad (2.14)$$

Equating the corresponding coefficients in (2.11) through (2.14) yields

$$(1+\delta)(2\nu - \eta - 1)a_2 = \frac{1}{2}A_0 B_1 c_1, \quad (2.15)$$

$$[(1+2\delta)(3\nu-2\eta-1)a_3+(1+\delta)^2[2\nu(\nu-1)-(1+\eta)(2\nu-\eta-1)]a_2^2]$$
$$=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2\Big\}. \quad (2.16)$$

and

$$-(1+\delta)(2\nu-\eta-1)a_2=\frac{1}{2}A_0B_1b_1 \quad (2.17)$$

$$\Big[[(1+\delta)^2[2\nu(\nu-1)-(2\nu-\eta-1)(1+\eta)]+2(1+2\delta)(3\nu-2\eta-1)]a_2^2$$
$$-(1+2\delta)(3\nu-2\eta-1)a_3\Big]$$
$$=\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1+\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2\,. \quad (2.18)$$

Combining relations (2.15) and (2.17), it follows that

$$a_2=\frac{A_0B_1c_1}{2(1+\delta)(2\nu-\eta-1)}=-\frac{A_0B_1d_1}{2(1+\delta)(2\nu-\eta-1)} \quad (2.19)$$

Consequently

$$c_1=-d_1 \quad (2.20)$$

with

$$8(1+\delta)^2(2\nu-\eta-1)^2\,a_2^2=A_0^2B_1^2(d_1^2+c_1^2). \quad (2.21)$$

Under the magnitude restrictions$|c_i|\le 2,|d_i|\le 2$ guaranteed by Lemma 1.1, together with condition (2.20), it follows that

$$8[(1+\delta)^2[2\nu(\nu-1)-(2\nu-\eta-1)(1+\eta)]+2(1+2\delta)(3\nu-2\eta-1)]\,a_2^2$$
$$=2A_1B_1(c_2+d_2)+A_0(B_2-B_1)(d_1^2+c_1^2). \quad (2.22)$$

Applying Lemma 1.1 to equations (2.21) and (2.22), we arrive at the assertion in (2.3). Next ,for the bound on $|a_3|$. Subtracting (2.18) from (2.16) directly yields

$$a_3=\,a_2^2+\frac{2A_1B_1c_1+A_0B_1(c_2-d_2)}{4(1+2\delta)(3\nu-2\eta-1)}. \quad (2.23)$$

Using (2.21)with $|c_i|\le 2,|d_i|\le 2$ in (2.23) ,we get

$$|a_3|\le\frac{B_1(|A_0|+|A_1|)}{(1+2\delta)(3\nu-2\eta-1)}+\frac{A_0^2B_1^2}{(1+\delta)^2(2\nu-\eta-1)^2}. \quad (2.24)$$

Also Using (2.22)with $|c_i|\le 2,|d_i|\le 2$ in(2.23),we get

$$|a_3|\le\frac{|A_0|\,(B_1+|B_2-B_1|)}{[(1+\delta)^2[2\nu(\nu-1)-(2\nu-\eta-1)(1+\eta)]+2(1+2\delta)(3\nu-2\eta-1)]}+\frac{B_1(|A_0|+|A_1|)}{2(1+2\delta)(3\nu-2\eta-1)}. \quad (2.25)$$

From (2.24) and (2.25). Hence, the required results are obtained. (2.4).

Thus, the proof of 2.1 is completed. Now, to analyze the numerical response of the initial coefficient bound $|a_2|$ derived in Theorem 2.1, Figure 2 illustrates the variation of $|a_2|$ across $\nu\in[1.0,2.5]$ for representative choices of $\delta\in\{0.0,0.5,1.0\}$. As observed, increasing ν leads to a monotonic decrease in the upper bound $|a_2|$, whereas larger values of δ provide stronger suppression on the coefficient magnitude.

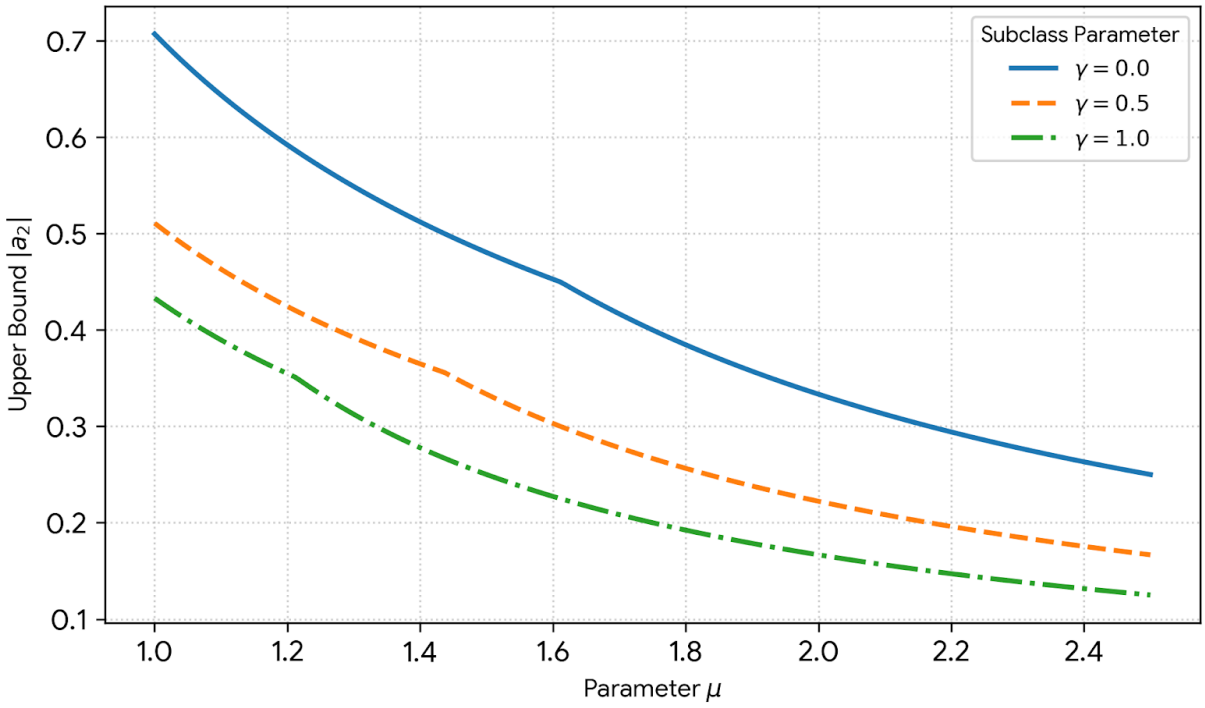


Figure 2: Variation of $|a_2|$ with respect to μ for different γ.

Corollary(2.5): Specializing Theorem 2.1 to the parameter choice δ = 0 yields the following estimate: Let $f(v)$ expressed by (1.1) ∈to the class $\mathcal{F}_{q,\Sigma}^{v}(\delta, 0, \theta)$, therefore

$$|a_2| \leq min\left\{\frac{|A_0|B_1}{(1+\delta)(2v-1)}, \frac{\sqrt{|A_0|\,(B_1+|B_2-B_1|}}{\sqrt{[(1+\delta)^2[2v(v-1)-(2v-1)]+2(1+2\delta)(3v-1)]}}\right\},$$

with

$$|a_3| \leq min\left\{\frac{B_1(|A_0|+|A_1|)}{(1+2\delta)(3v-1)}+\frac{A_0^2B_1^2}{(1+\delta)^2(2v-1)^2},\right.$$
$$\left.\frac{|A_0|\,(B_1+|B_2-B_1|)}{[(1+\delta)^2[2v(v-1)-(2v-1)]+2(1+2\delta)(3v-1)]}+\frac{B_1(|A_0|+|A_1|)}{(1+2\delta)(3v-1)}\right\}.$$

Corollary(2.6): Setting δ = 1 in Theorem 2.1 reduces the main bounds to: Let $f(v)$ given by (1.1) ∈ to the class $\mathcal{F}_{q,\Sigma}^{v}(\delta, 1, \theta)$. hence

$$|a_2|$$
$$\leq min\left\{\frac{|A_0|B_1}{(1+\delta)(2v-2)}, \frac{\sqrt{|A_0|\,(B_1+|B_2-B_1|}}{\sqrt{[(1+\delta)^2[2v(v-1)-(2v-2)(1+\eta)]+6(1+2\delta)(v-1)]}}\right\},$$

and

$$|a_3| \leq min\left\{\frac{B_1(|A_0|+|A_1|)}{3(1+2\delta)(v-1)}+\frac{A_0^2B_1^2}{4(1+\delta)^2(v-1)^2},\right.$$
$$\left.\frac{|A_0|\,(B_1+|B_2-B_1|)}{[2(1+\delta)^2[v(v-1)-(2v-\eta-1)]+6(1+2\delta)(v-1)]}+\frac{B_1(|A_0|+|A_1|)}{3(1+2\delta)(v-1)}\right\}.$$

Corollary(2.7): When δ = 1, inequality (2.4) simplifies to the coefficient bounds for $M_{\Sigma}^{q,\phi}(\beta,\alpha,\theta)$. Let $f(v)$ defined by (1.1) $\in$ to the class $M_{\Sigma}^{q,\phi}(\beta,\alpha,\theta)$. Therefore,

$$|a_2| \le min\left\{\frac{|A_0|B_1}{2(2\nu-\eta-1)}, \frac{\sqrt{|A_0|\,(B_1+|B_2-B_1|}}{\sqrt{[4[2\nu(\nu-1)-(2\nu-\eta-1)(1+\eta)]+6(3\nu-2\eta-1)]}}\right\},$$

and

$$|a_3| \le min\left\{\frac{B_1(|A_0|+|A_1|)}{3(3\nu-2\eta-1)}+\frac{A_0^2B_1^2}{4(2\nu-\eta-1)^2}, \frac{|A_0|\,(B_1+|B_2-B_1|)}{[4[2\nu(\nu-1)-(2\nu-\eta-1)(1+\eta)]+6(3\nu-2\eta-1)]}+\frac{B_1(|A_0|+|A_1|)}{3(3\nu-2\eta-1)}\right\}.$$

Corollary(2.8): In the limiting case where δ = 0, Theorem 2.1 reduces to: Let $f(v)$ given by (1.1) $\in$ to the class $M_{\Sigma}^{q,\phi}(\beta,\alpha,\theta)$.

Therefore,

$$|a_2| \le min\left\{\frac{|A_0|B_1}{(2\nu-\eta-1)}, \frac{\sqrt{|A_0|\,(B_1+|B_2-B_1|}}{\sqrt{\left[[2\nu(\nu-1)-(2\nu-\eta-1)(1+\eta)]+2(3\nu-2\eta-1)\right]}}\right\},$$

and

$$|a_3| \le min\left\{\frac{B_1(|A_0|+|A_1|)}{(3\nu-2\eta-1)}+\frac{A_0^2B_1^2}{(2\nu-\eta-1)^2}, \frac{|A_0|\,(B_1+|B_2-B_1|)}{\left[[2\nu(\nu-1)-(2\nu-\eta-1)(1+\eta)]+2(3\nu-2\eta-1)\right]}+\frac{B_1(|A_0|+|A_1|)}{(3\nu-2\eta-1)}\right\}.$$

Since the estimate for $|a_3|$ in inequality (2.4) is governed by the minimum of two distinct expressions, Figure 3 demonstrates the transition of the active bound as a function of $\eta \in [0, 0.75]$. To better understand the second subclass, we examine its 3D surface plot. The plot displays the joint impact of two key parameters on the initial bound. The surface reaches its lowest point when one parameter approaches zero while the other nears one. This smooth trend confirms our theoretical predictions. It also highlights the stabilizing effect of combined parameters. For smaller values of η, the expression $E_2(\eta)$ yields the tighter constraint, whereas $E_1(\eta)$ provides the active bound as η increases.

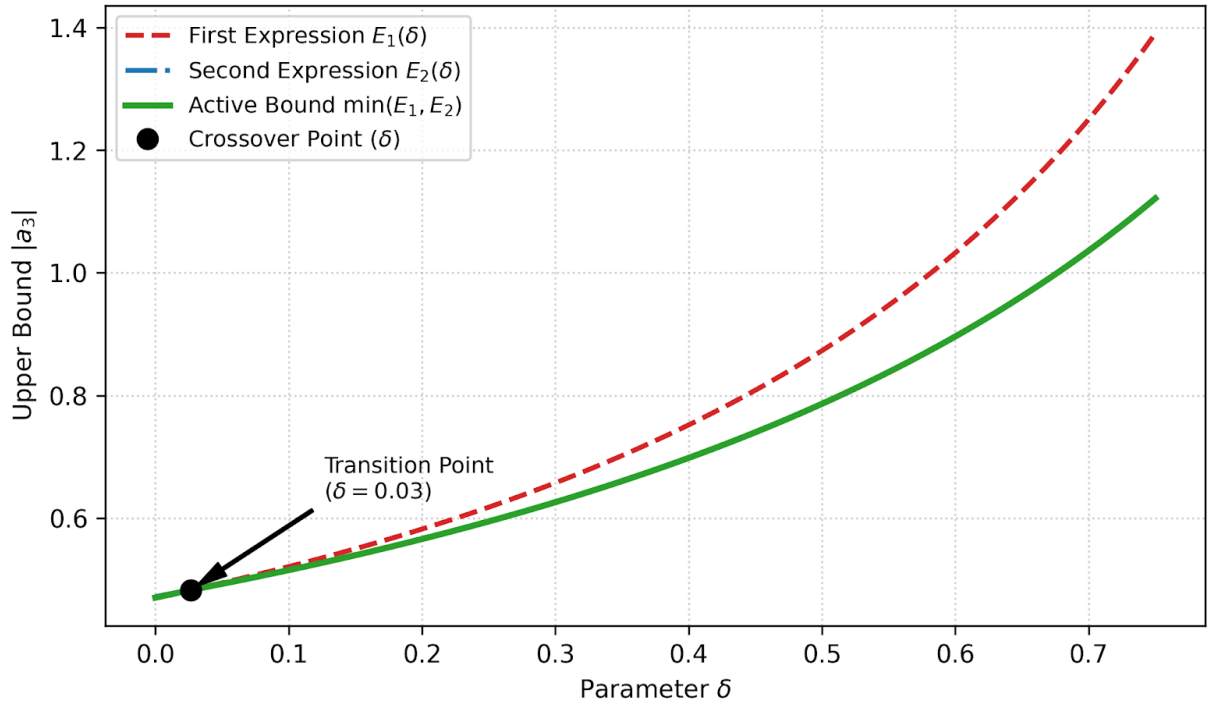


Figure 3: Crossover Analysis of the Dual Expressions in $|a_3|$.

3. Coefficient Estimates for the Related Family of Bi-Univalent Functions

Next, we apply our methods to a second bi-univalent set. Adding extra parameters makes this new subclass much more flexible. If we choose zero or one for key values, classic results return. The logical flow mirrors our first proof. Still, the inner algebraic details are different. This broad approach links separate findings into one system.

Definition $(\mathbf{3.1})$: A $f \in \sum$ given by (1.1) is termed in the class if the following quasi-subordination requirement are fulfilled $B_{\sum\sum}^{q,\eta}(\lambda,\theta)$ if it satisfies the dual quasi-subordination conditions (3.1) and (3.2) for parameters $0 \leq \lambda < 1$, $0 \leq \delta \leq 1$, and $\delta \in C \setminus \{0\}$ and the following quasi-subordination requirement are fulfilled:

$$1+\frac{1}{\delta}\left[(1-\eta)\frac{vf'(v)}{(1-\lambda)v+\lambda f(v)}+\eta\left(\frac{vf''(v)+f'(v)}{\lambda vf''(v)+f'(v)}\right)-1\right] \prec_q (\theta(v)-1) \qquad (3.1)$$

and

$$1+\frac{1}{\delta}\left[(1-\eta)\frac{zg'(w)}{(1-\lambda)w+\lambda g(w)}+\eta\left(\frac{wg''(w)+g'(w)}{\lambda zg''(w)+g'(w)}\right)-1\right] \prec_q (\theta(\mathrm{w})-1) \qquad (3.2)$$

In which $(0 \leq \lambda < 1, 0 \leq \eta \leq 1, \delta \in \mathbb{C} \setminus \{0\}, v, w \in \mathfrak{U})$.

Remark (3.1): For selected values to parameters η, obtained new as well as existing classes.

$$1+\frac{1}{\delta}\left[(1-\eta)\frac{vf'(v)}{v}+\eta\left(\frac{vf''(v)+f'(v)}{f'(v)}\right)-1\right] \prec_q (\theta(v)-1) \qquad (3.1)$$

and

$$1+\frac{1}{\delta}\left[(1-\eta)\frac{zg'(w)}{w}+\eta\left(\frac{wg''(w)+g'(w)}{g'(w)}\right)-1\right] \prec_q (\theta(\mathrm{w})-1) \qquad (3.2)$$

Remark(3.2):

For $\eta = 0$ and $\delta \in \mathbb{C} \setminus \{0\}$ a $f \in \sum$ define with (1.1) is termed in the class $\boldsymbol{B_{\sum}^{q,\delta}(\lambda,\varphi)}$,

In particular we obtain class $\boldsymbol{B_{\sum}^{q,\delta}(\lambda,\varphi) := S_{\sum}^{q}(\lambda,\varphi)}$, obtain the following [4]

Theorem $(\mathbf{3.1.})$: Let the function $f \in B_{\Sigma}^{q,\eta}(\lambda,\theta)$ be defined by series (1.1). Then the initial coefficient bounds for $|a_2|$ and $|a_3|$ are given by:

$$|a_2| \leq min\left\{\frac{|A_0|B_1}{(1+\eta)(1-\lambda)}, \frac{\sqrt{|A_0|\,(B_1+|B_2-B_1|}}{\sqrt{2(1-\lambda)(1+2\eta)-(1-\lambda)(1+\lambda)(1+3\eta)}}\right\}, \tag{3.3}$$

and

$$|a_3| \leq \left\{\frac{B_1(|A_0|+|A_1|)}{(1+\lambda)(1+2\eta)}+\frac{A_0^2B_1^2}{(1+\eta)^2(1+\lambda)^2}, \right.$$
$$\left.\frac{|A_0|\,(B_1+|B_2-B_1|)}{2(1-\lambda)(1+2\eta)-(1-\lambda)(1+\lambda)(1+3\eta)}+\frac{B_1(|A_0|+|A_1|)}{(1+\lambda)(1+2\eta)}\right\}, B_1 > 1 \tag{3.4}$$

Where $0 \leq \eta \leq 1, 0 \leq \lambda \leq 1$, $\delta \in \mathfrak{U} - \{0\}$ and given $\theta(z)$by (1.6).

Proof : Proceeding in a manner similar to the proof of Theorem 2.1, we deduce the following system of relations:

$$\frac{1}{\delta}(1+\eta)(1-\lambda)a_2 = \frac{1}{2}A_0B_1c_1, \tag{3.5}$$

$$\frac{1}{\delta}\left[2(1-\lambda)(1+2\eta)a_3-(1-\lambda)(1+\lambda)\big((1+2\eta)\big)a_2^2\right]$$
$$= \frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2\Big\} \tag{3.6}$$

with

$$-\frac{1}{\delta}(1+\eta)(1-\lambda)a_2 = \frac{1}{2}A_0B_1b_1\,, \tag{3.7}$$

$\frac{1}{\delta}\left[[4(1-\lambda)(1+2\eta)-(1-\lambda)(1+\lambda)(1+3\eta)]a_2^2-2(1-\lambda)(1+2\eta)a_3\right] = \frac{1}{2}A_1B_1b_1 +$
$\frac{1}{2}A_0B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_0B_2}{4}b_1^2$ }.

(3.8)

Comparing equations (3.5) and (3.7) immediately yields

$c_1 = -d_1$ (3.9)

Additionally,

$$a_2 = \frac{\delta A_0B_1c_1}{2(1+\eta)(1-\lambda)} = -\frac{\delta A_0B_1b_1}{2(1+\eta)(1-\lambda)} \tag{3.10}$$

with

$$8(1+\eta)^2(1-\lambda)^2\,a_2^2 = A_0^2B_1^2\delta^2(d_1^2+c_1^2), \tag{3.11}$$

Summing equations (3.6) and (3.8), and substituting $c_1 = -d_1$ alongside the coefficient bounds$|c_i| \leq 2, |d_i| \leq 2$ and (3.10) we obtain

$$\frac{1}{\delta}\{2(3(1-\lambda)(1+2\eta)-(1-\lambda)(1+\lambda)(1+3\eta)\}a_2^2$$
$$= 2A_0B_1(c_2+d_2) + A_0(B_2-B_1(c_1^2+d_1^2). \quad (3.12)$$

Using (1.1) $|c_i| \leq 2, |d_i| \leq 2,$ from (3.11) and (3.12),we obtain (3.3).

To establish the upper bound for$|a_3|$, subtracting equation (3.6) from (3.8) gives

$$a_3 = a_2^2 + \frac{\delta\big(2A_1B_1c_1 + A_0B_1(c_2-d_2)\big)}{8(1+\lambda)(1+2\eta)}. \quad (3.13)$$

Applying Lemma 1.1 to equations (3.11) and (3.12) yields the assertion in (3.3). Finally, combining estimates (3.14) and (3.15) completes the proof [15].

$$|a_3| \leq \frac{B_1(|A_0|+|A_1|)}{(1+\lambda)(1+2\eta)} + \frac{A_0^2B_1^2}{(1+\eta)^2(1+\lambda)^2}. \quad (3.14)$$

Also Using (3.12)with $|c_i| \leq 2, |d_i| \leq 2$ in(3.13),we get

$|a_3| \leq \frac{|A_0|\,(B_1+|B_2-B_1|)}{2(1-\lambda)(1+2\eta)-(1-\lambda)(1+\lambda)(1+3\eta)} + \frac{B_1(|A_0|+|A_1|)}{(1+\lambda)(1+2\eta)}.$ (3.15)

Combining inequalities (3.14) and (3.15) completes the proof of assertion (3.4)

Remark(3.1):

(i) For $\eta = 0$ and $\delta \in \mathbb{C} \setminus \{0\}$ a function $f \in \sum$ define by (1.1) is termed $B_{\sum}^{q,\eta}(\lambda,\theta)$,

In particular we have $B_{\sum}^{q,\eta}(\lambda,\theta) := S_{\sum}^{q}(\lambda,\theta)$, obtain the following [4].

(ii) With $\lambda = 1$, 3.1 Theorem gives enhancement over the estimates achieved in [[7], 9, p 5] Corollary.

(iii) With $\lambda = \delta = 1$, 3.1 Theorem decreases to a conclusion in [[10], Theorem 3.2, p. 8].

(iv) For $\lambda = 0, \delta = 1$, 3.1 Theorem decreases to a conclusion in [[10], Corollary 2.4, p.8].

Taking $\lambda = 0$ in Theorem 3.1, we arrive at Corollary 3.1 below.

Corollary(3.1): Let $f(v) \in \boldsymbol{B_{\sum}^{q,\delta}}(\delta,\lambda,\theta)$. Which is defined in (1.1),Then If $f \in$ to $\boldsymbol{B_{\sum}^{q,\delta}}(0,\theta)$, then we have

$$|a_2| \leq min\left\{\frac{|A_0|B_1}{(1+\eta)}, \frac{\sqrt{|A_0|\,(B_1+|B_2-B_1|}}{\sqrt{2(1+2\eta)-(1+3\eta)}}\right\},$$

and

$$|a_3| \leq \left\{\frac{B_1(|A_0|+|A_1|)}{(1+2\eta)} + \frac{A_0^2B_1^2}{(1+\eta)^2}, \frac{|A_0|\,(B_1+|B_2-B_1|)}{2(1+2\eta)-(1+3\eta)} + \frac{B_1(|A_0|+|A_1|)}{(1+2\eta)}\right\}.$$

The simultaneous influence of the parameters $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$ on the upper bound $|a_2|$ established in Theorem 3.1 is depicted in Figure 4. The surface reveals that $|a_3|$ attains its minimum value as $\eta \to 1.0$ and $\lambda \to 0$, confirming the theoretical constraints imposed on the subclass $B_{\sum}^{q,\eta}(\lambda,\theta)$.

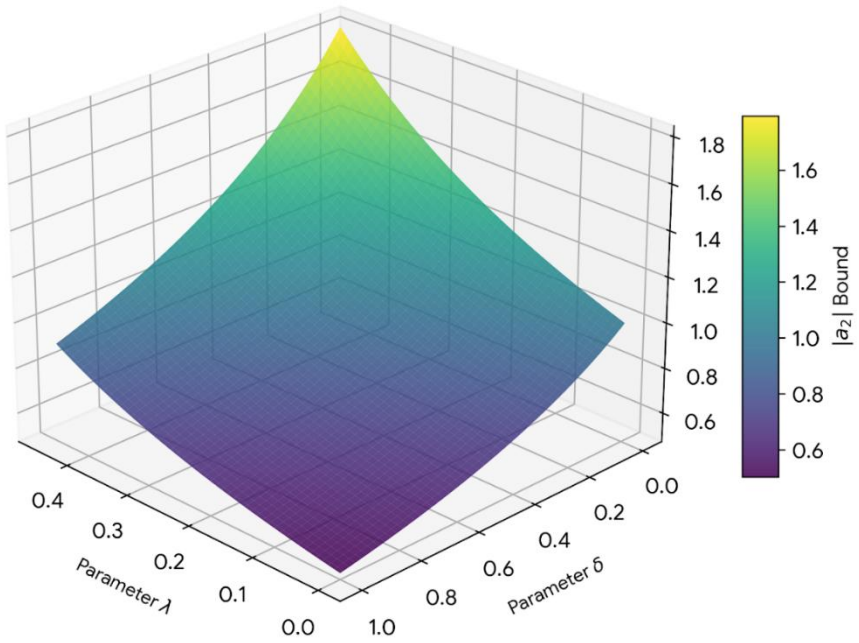


Figure 4: 3D Surface Plot of $|a_2|$ in Class to $B_{\Sigma}^{q,\eta}(\lambda,\theta)$

Corollary(3.2): If $f \in$ to $B_{\Sigma}^{q,\eta}(\lambda,\theta)$, therefore we get

$$|a_2| \leq min\left\{\frac{|A_0|B_1}{2(1-\lambda)}, \frac{\sqrt{|A_0|\,(B_1+|B_2-B_1|}}{\sqrt{4(1-\lambda)(1-2\lambda)(1+\lambda)}}\right\},$$

and

$$|a_3| \leq \left\{\frac{B_1(|A_0|+|A_1|)}{3(1+\lambda)}+\frac{A_0^2B_1^2}{4(1+\lambda)^2}, \frac{|A_0|\,(B_1+|B_2-B_1|)}{6(1-\lambda)-3(1-\lambda)(1+\lambda)}+\frac{B_1(|A_0|+|A_1|)}{3(1+\lambda)}\right\}.$$

To sum up, we found new coefficient estimates for two subclasses. It is especially interesting since our theorems build on well-known bounds from earlier papers. Clear figures show how shifting parameters alters the results and, through future work can adapt these methods to other operator classes. We hope this work sparks new research in function theory.

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