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Fuzzy Frame Domination of Strong Fuzzy Graphs

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ABSTRACT

Throughout this work, the new domination parameter which is called fuzzy frame domination is introduced. Some important properties and bounds of this parameter are been clarified. Also, determined this number to the standard graphs and it is a complement. Moreover, the effect of deleting a vertex or an edge and add an edge are being discussed

1. Introduction

The development of research into the concept of the graph theory lead to the emergence of a new concept called fuzzy graph. The first lay the foundation for fuzzy set theory and fuzzy relationships was zadeh in 1965[1]. In 1975, Rosenfeld published his research paper and he is consider the true founder having provided the basic definition fuzzy graph [2]. This concept was based on the development of comprehensive mathematical formulas that describe the general properties of fuzzy graph. Assume $G = (V, E)$ be undirected , finite, simple graph. A fuzzy graph is characterized as $G = (\sigma, \mu)$ where $\mu: V \times V \to [0,1]$ and $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ and V is a non-empty collection of vertices known as a fuzzy subset. If $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, then an edge (u, v) is considered effective. The fuzzy graph is considered strong if every edge that is belong to it is effective. $N_E(v) = \{u; (u, v)\ \text{i an effective edge}\}$ and $N_E[v] = N_E(v) \cup \{v\}$are the open effective neighborhoods. Continued research on the concept of domination, introduced in 1963, led to the development of the concept frame domination. The first paper on the frame domination was published in 2017 by A. Omran, and Rajihy [3], and further research on the frame domination was developed in 2020 by Sarai Hamza, A. Omran and Rajihy [4-5]. This paper introduces a new definition of frame domination called a fuzzy frame domination, which deal with strong fuzzy graph. It examines some basic properties and the changes that occur when a vertex and an edge are removed, as well as when an edge is added for stander graph.

2. Fuzzy Frame Domination in Graph

In this section study a new type of domination say that Fuzzy Frame Domination in Graph.

Definition 2.1 Let $G = (\sigma, \mu)$ be a strong fuzzy graph, *a set* $F \subseteq V(G)$ is known a fuzzy frame dominating set of G if F is frame dominating set.

Definition 2.2 The minimum cardinality of a fuzzy frame dominating set of G is known the a fuzzy frame domination number and is indicated by $\gamma_{Ff}(G)$.

Example 2.3 Let $G = (V, E)$ be a strong fuzzy graph the sets F_i is a frame dominating sets.

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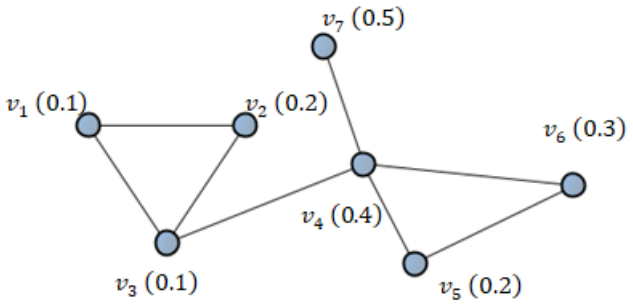


Figure 1: FFDS

$F_1 = \{v_1, v_4, v_7\} = 0.1$, $F_2 = \{v_1, v_5, v_7\} = 0.8$, $F_3 = \{v_1, v_6, v_7\} = 0.9$, $F_4 = \{v_2, v_4, v_7\} = 0.11$, $F_5 = \{v_2, v_5, v_7\} = 0.9$, $F_6 = \{v_2, v_6, v_7\} = 0.10$, $F_7 = \{v_3, v_4, v_7\} = 0.10$, $F_8 = \{v_3, v_5, v_7\} = 0.8$, $F_9 = \{v_3, v_6, v_7\} = 0.9$. Hence, F_2 andF_8 is the minimum fuzzy frame dominating set and $\gamma_{Ff}(G) = 0.8$

Remark 2.4: Every vertex not belonging to whichever cycle is belongs to any fuzzy frame dominating set.

Observation 2.5: Let $G = (\sigma, \mu)$ be a strong fuzzy graph, then:

1. $\gamma_{Ff}(C_n) = min\,\sigma(v_i) = \begin{cases} \gamma_{Ff}(W_n) & if\ n \geq 3 \\ \gamma_{Ff}(K_n) & if\ n \geq 3 \\ \gamma_{Ff}(K_{n,m}) & if\ n\ , m \geq 2 \end{cases}$
2. $\gamma_{Ff}(P_n) = \sum_{i=1}^{n} \sigma(v_i) = \begin{cases} \gamma_{Ff}(N_n) \\ \gamma_{Ff}(S_n) \end{cases}$
3. $\gamma_{Ff}(K_n) = \sum_{i=1}^{n} \sigma(v_i)$, if $n\ or\ m = 1$.
4. $\gamma_{Ff}(K_{n,m}) = \sum_{i=1}^{n+m} \sigma(v_i)\ n\ and\ m \geq 2$.
5. $\gamma_{Ff}(G) = \sum_{i=1}^{n} \sigma(v_i)$, when G is a forest.

Proposition 2.6: Let $G = (\sigma, \mu)$ is a strong fuzzy graph :

1. If G has a fuzzy frame domination number, then $min\,\sigma(v_i) \leq \gamma_{Ff}(G) \leq \sum_{i=1}^{n} \sigma(v_i)$.
2. If G is an Eulerian graph, then G has a fuzzy frame domination number and $\gamma_{Ff}(G) = min\,\sigma(v_i)$.
3. If G is a Hamilton graph, then G has a fuzzy frame domination number and $\gamma_{Ff}(G) = min\,\sigma(v_i)$.
4. If G has a fuzzy frame domination number and its disconnected graph with H components, then $\gamma_{Ff}(G) = \sum_{i=1}^{n} \sigma(v_i)$.

Proof:

1. It clear from observation 2.5.
2. And 3. It clear from observation 2.5 (1).

4. when G fuzzy graph contains two or more component than fuzzy graphs (H, K, R, M) has fuzzy frame domination number. Then $\gamma_{Ff}(G) = \gamma_{Ff}(H) + \gamma_{Ff}(K) + \gamma_{Ff}(R) + \gamma_{Ff}(M)$, hence $\gamma_{Ff}(G) = \sum_{i=1}^{n} \sigma(v_i)$.

3. Some complement of graph

proposition 3.1: The fuzzy frame domination number of a part of complement special graphs

1) $\gamma_{Ff}(\overline{C_n}) = \begin{cases} \sum_{i=1}^{n} \sigma(v_i) & if\ n = 3,4 \\ min\, \sigma(v_i) & if\ n \geq 5 \end{cases}$
2) $\gamma_{Ff}(\overline{P_n}) = \begin{cases} \sum_{i=1}^{n} \sigma(v_i) & if\ \ n \leq 4 \\ min\, \sigma(v_i) & if\ n \geq 5 \end{cases}$
3) $\gamma_{Ff}(\overline{W_n}) = \begin{cases} \sum_{i=1}^{n} \sigma(v_i) & if\ n = 4,5 \\ \sigma(v_1) + min\, \sigma(v_{i-1}) & if\ n \geq 6 \end{cases}$
4) $\gamma_{Ff}(\overline{K_{n,m}}) = \begin{cases} min\, \sigma(v_i) & if\ n, m \geq 3 \\ \sum_{i=1}^{n} \sigma(v_i) & if\ n, m = 2 \\ \sum_{i=1}^{n} \sigma(v_n) + min\, \sigma(v_m) & if\ \ n = 1,2\ , m \geq 3\ or\ (n \geq 3, m = 1,2\ ) \end{cases}$
5) $\gamma_{Ff}(\overline{S_n}) = \begin{cases} \sum_{i=1}^{n} \sigma(v_i) & if\ \ n \leq 3 \\ \sigma(v_1) + min\, \sigma(v_i); i = 1, \dots, n-1, where\ v_1\ is\ the\ center\ of\ star & if\ n \geq 4 \end{cases}$

Proof: Let $G = (V, E)$ be a fuzzy frame domination number, as following discuses some complement graph:

1) If $G = \overline{C_n}$, we have to case depend on order of cycle:
   i. If $n = 3$, then $\overline{C_n} \cong N_n$ all vertex belong to fuzzy frame dominating set. According to properties 2.5 (1), $\gamma_{Ff}(\overline{C_3}) = \gamma_{Ff}(N_3) = \sum_{i=1}^{3} \sigma(v_i)$.
   ii. If $n = 4$,then $\overline{C_4} \cong P_2 \cup P_2$.According to properties 2.5(2), $\gamma_{Ff}(\overline{C_4}) = \gamma_{ff}(P_2)\ and\ \gamma_{ff}(P_2) = \sum_{i=1}^{4} \sigma(v_i)$.
   iii. If $n \geq 5$, then $\overline{C_n}$ is a cycle according to observation 2.5(1), $\gamma_{Ff}(\overline{C_n}) = min\, \sigma(v_i)$.

2) If $G = \overline{P_n}$, then there are two cases:

   i. If $n \leq 4$, then $\overline{P_n} \cong \begin{cases} N_2 & when\ \ n = 2 \\ P_2\ and\ isolated\ vertex\ (N_1) & when\ n = 3 \\ P_4 & when\ n = 4 \end{cases}$

   In this case the $\gamma_{Ff}(\overline{P_{n\leq 4}}) = \sum_{i=1}^{n} \sigma(v_i)$.

   ii. If n ≥ 5, then $\overline{P_n} \cong C_n$. According to observation 2.5(1), $\gamma_{Ff}(\overline{P_n}) = \gamma_{Ff}(C_n) = min\, \sigma(v_i)$.

3) If $G = \overline{W_n}$, there are three case :
   i. If $n = 4$, then $\overline{W_n} \cong N_n$. Hence, $\gamma_{Ff}(\overline{W_n}) = \gamma_{Ff}(N_n) = \sum_{i=1}^{4} \sigma(v_i)$.
   ii. If $n = 5$, then $\overline{W_n} \cong P_2 \cup P_2 \cup N_1$,then $\gamma_{Ff}(\overline{W_n}) = \gamma_{Ff}(N_1) + \gamma_{Ff}(P_2) + \gamma_{Ff}(P_2) = \sum_{i=1}^{5} \sigma(v_i)$.
   iii. If n ≥ 6,then $\overline{W_n} = C_n + N_1$.Thus,$\gamma_{Ff}(\overline{W_n}) = \gamma_{Ff}(C_n) + \gamma_{Ff}(N_1) = \sigma(v_1) + min\, \sigma(v_{i-1})$.
4) If $G = \overline{K_{n,m}}$, the complete bipartite is divided to two set of the vertex. Then discuss two cases as following:
   i. If n, m ≥ 3, In this case, every set contains complete graph say K_n and K_m. Therefor $\gamma_{Ff}(\overline{K_{n,m}}) = \gamma_{Ff}(K_n) + \gamma_{Ff}(K_m) = min\, \sigma(v_i)$.
   ii. If $n, m = 2$, the two sets convert to disconnected graph that contains two path. Then $\gamma_{ff}(\overline{K_{n,m}}) = \gamma_{Ff}(P_n) + \gamma_{Ff}(P_m) = \sum_{i=1}^{n} \sigma(v_i)$.

iii. If $n = 1 or\ 2, m \geq 3$ or $(n \geq 3, m = 1 or\ 2)$, the two sets convert to disconnected graph contains null graph of order 1or path of order 2 and complement $N_1 \cup (K_m)$. Therefore $\gamma_{Ff}(\overline{K_{n,m}}) = \gamma_{Ff}(N_1) + \gamma_{Ff}(K_m) = min\,\sigma(v_i) + \sum_{i=1}^{n} \sigma(v_i)$.

5) If $G = S_n$, there are two case discuss as following:

i. If $n \leq 3$, then $\overline{S_n} \cong \begin{cases} N_2 & when\ n = 2 \\ P_2 \cup N_1 & when\ n = 3 \end{cases}$
According to properties 2.5, $\gamma_{Ff}(\overline{S_n}) = \sum_{i=1}^{n} \sigma(v_i)$.

ii. If $n \geq 4$, then $\overline{S_n} \cong K_{n-1} \cup N_1$. Hence, $\gamma_{Ff}(\overline{S_n}) = \gamma_{Ff}(K_{n-1}) + \gamma_{Ff}(N_1) = \sum_{i=1}^{n-1} \sigma(v_1) + min\,\sigma(v_i); i = 1, \dots, n-1, where\ v_1\ is\ the\ center\ of\ star$.

4. Change and unchanged

Theorem 4.1: Let G be a strong fuzzy graph has FFDS

i. If G contain a cycle, then $\gamma_{Ff}(G - v) \geq \gamma_{Ff}(G)$.
ii. If G has no cycle, then $\gamma_{Ff}(G - v) < \gamma_{Ff}(G)$.

Proof: Let $G = (\sigma, \mu)$ be a fuzzy frame graph and v is minimum fuzzy frame dominating set . I will discuss two different cases in the proof, as illustrated in the cases below:

i. When G contain a cycle and v belong one cycle or more we discuss four case:

Case 1: If G is a cycle for order $n \geq 3$. then either v belong to one cycle of order $n \geq 3$, then $G - v$ represent a path of order $n - 1$ or v belong to two or cycles of order $n \geq 4$,then $G - v$ became a cycle with end vertex. Therefore $\gamma_{Ff}(G - v) > \gamma_{Ff}(G)$.

Case 2: If G graph has a common vertex (v). Then $G - v$ became at least two path of order $n \geq 2$. Hence $\gamma_{Ff}(G - v) > \gamma_{Ff}(G)$.

Case 3: If G be a graph has a bridge between cycles, $G - v$ became cycle with path. Therefore $\gamma_{Ff}(G - v) > \gamma_{Ff}(G)$.

Case 4: If G is a cycle and v belong to more cycle. This cases dependent on $\sigma(v)$:
1. If $\sigma(v_i)$ equal to another vertex in G, then $\gamma_{Ff}(G - v) = \gamma_{Ff}(G)$.
2. If $\sigma(v_i)$ is different in G, therefore $\gamma_{Ff}(G - v) > \gamma_{Ff}(G)$.

ii. If G is null or tree, then either v belong to null graph of order $n \geq 2$, hence $G - v$ still a null graph of order $n - 1$ or v belong to path of order $n \geq 2$ then $G - v$ either isolated vertex or path with end vertex. Therefore $\gamma_{Ff}(G - v) < \gamma_{Ff}(G)$.

Theorem 4.2: Let $G = (\sigma, \mu)$ be a strong fuzzy graph (path or cycle or G has bridge) and has $\gamma_{Ff}(G)$ then $\gamma_{Ff}(G - e) \geq \gamma_{Ff}(G)$.
Proof: Four cases are being discussed as the following:
Case 1: If G be a cycle of order $n \geq 3$, then $G - e$ developed into path of order $n \geq 2$
from observation 2.4, $\gamma_{Ff}(G - e) > \gamma_{Ff}(G)$.

Case 2:if G contain two cycle connected common edge, at least one of them has cycle of order 3. Now discus there sub cases as following:

i. Subcase1: If e common edge, then $G-e$ became a cycle of same order. Hence $\gamma_{Ff}(G-e)=\gamma_{Ff}(G)$.

ii. Subcase2: if e belong to cycle of order $n=3$, then $G-e$ developed into cycle with end vertex. Consequently $\gamma_{Ff}(G-e)>\gamma_{Ff}(G)$.

iii. If e belong to cycle of order $n\geq 4$, then $G-e$ developed into a cycle of order $n=3$with path. Consequently $\gamma_{Ff}(G-e)>\gamma_{Ff}(G)$.

Case 3: If G is a connected graph and e is a bridge between to compotes, then $G-e$ change to disconnected graph. Therefore $\gamma_{Ff}(G-e)=\gamma_{Ff}(G)$.

Case 4: If G be a path of order $n\geq 2$, then G − e developed into a null graph of order 2 or isolated vertex with path of order $n-1$. Hence $\gamma_{Ff}(G-e)=\gamma_{Ff}(G)$.

Theorem 4.3: Let $G=(\sigma,\mu)$ be a strong fuzzy graph (cycle or path or null or has only two components or cycle with leaf) and has $\gamma_{Ff}(G)$, then $\gamma_{Ff}(G+e)\leq\gamma_{Ff}(G)$.

Proof:

Csae1: Let G be a cycle of order $n\geq 4$.Then $G+e$ became a two cycles disconnected him by an edge. Therefore $\gamma_{Ff}(G+e)=\gamma_{Ff}(G)$.

Case 2: Let G be a path of order $n\geq 4$.Then $G+e$ change to a cycle of same order $\gamma_{Ff}(G+e)<\gamma_{Ff}(G)$.

Case 3: Let G be a null graph; as following disease two sub case :

Subcase1: If $n=2$. Then $G+e$ change to path of same order. Hence $\gamma_{Ff}(G+e)=\gamma_{Ff}(G)$.

Subcase 2: If $n>2$. Then $G+e$ became isolated vertex with a path of order 2, therefore $\gamma_{Ff}(G+e)=\gamma_{Ff}(G)$.

Case 4: Let G be two components. Then $G+e$ developed to a connected graph by a bridge (e). consequently $\gamma_{Ff}(G+e)=\gamma_{Ff}(G)$.

Case 5: Let G be a cycle with leaf. Then $G+e$ change to a cycle. Hence $\gamma_{Ff}(G+e)<\gamma_{Ff}(G)$.

5. Conclusions

In this investigate, the basic concept of fuzzy frame domination has been extended to different forms of graphs. Many new properties appeared, and the calculations also differed in standard graphs. There were also many differences in the extent to which the new frame domination number was affected by deleting a vertex, deleting or adding an edge compared to the calculations of these effects on the FFDS.

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