Spectral Analysis of Self-Adjoint Fractional Sturm-Liouville Operators in Weighted Hilbert Spaces

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ABSTRACT

In this paper, we establish a comprehensive spectral theory for a class of fractional Sturm-Liouville differential operators defined on weighted hilbert spaces $L_2\ (a, b\ ; w)$. Incorporating both Riemann- Liouville and Caputo fractional derivatives, we construct a well-defined inner product space and prove the self-adjointness of the modified fractional differential operator under suitable non-local boundary conditions. Using functional-analytic methods, including the construction of a symmetric fractional Green's function, we demonstrate that the resolvent operator is compact and self-adjoint. Consequently, we establish the existence of a orthonormal basic of eigenfunctions in $L_2\ (a, b\ ; w)$. Finally, we provide variational characterizations and asymptotic estimates for the spectrum.

1. Introduction

Fractional calculus-the study of integrals and derivatives of non-integer orders-has emerged as a powerful mathematical tool for modeling non-local, memory-dependent, and anomalous diffusive phenomena across diverse scientific and engineering disciplines [1]. Traditional integer-order differential operators fail to capture long-range temporal or spatial correlations adequately, whereas fractional differential operators inherently account for spatial non-locality [2].

In classical differential equations, the Sturm-Liouville theory forms the bedrock of spectral analysis, separation of variables, and expansions in orthogonal functions [3]. Extending integer-order Sturm-Liouville theory to fractional differential operators present severe functional-analytic challenges due to non-local nature of fractional derivatives, non-commutativity, and the non-self-adjointness of standard fractional differential operators under classical boundary conditions [4]. Recent advances have focused on constructing modified fractional differential operators that preserve self-adjointness in weighted Hilbert spaces [5]. Weight functions are essential in mathematical physic and dynamic modeling. As they account for inhomogeneous densities and non-uniform measure spaces [6].

The primary objective of this paper is to formulate a self-adjoint fractional Sturm-Liouville operator in a weighted Hilbert space $L_2\ (a, b\ ; w)$ and perform a complete spectral analysis. We develop the domain of the operator, establish the symmetry of its inverse via a fractional Green's function, prove compactness of the resolvent operator, and demonstrate the completeness of its eigenfunctions.

2. Statement of The Problem

Let $[a, b] \subset \mathbb{R}$ be a bounded interval, and let $w(x) > 0$ almost everywhere on $[a, b]$ be a positive weight function such that $w \in L_1\ (a, b)$ and $(1/w) \in L\infty(a, b)$.

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We consider the weighted Hilbert space $\mathcal{H} = L_2\ (a, b\ ; w)$ equipped with the weighted inner product:

$$< f, g >_w = \int_a^b f(x)\ .\ \bar{g}(x)\ .w(x)\ dx$$

and the associated norm $||f||_w = \sqrt{(< f, f >_w)}$.

Consider the fractional Sturm-Liouville differential expression given by:

$$L_\alpha u(x) = (1/w(x))\ .[-D_b^{-a}\ (p(x).D_a^{+a}\ u(x)) + q(x)\ .u\ (x)]\ , x \in (a, b)$$

Where $0.5 < a \leq 1. p(x) > 0, p \in C^1[a, b]\ , q(x) \geq 0, p \in C[a, b]$, and D_a^{+a}, D_b^{-a} denote the left-Caputo and right-Riemann-Liouville fractional derivatives of order a, respectively.

Research Problem: Standard definitions of fractional derivatives lack skew-symmetry under standard integration by parts, leading to non-self-adjoint operators in $L_2\ (a, b)$. The main open challenge addressed in this paper is [جديد]:

1. Constructing an appropriate domain $Dom(T) \subset L_2\ (a, b\ ; w)$ with non-local boundary conditions under which $Tu = L_au$ is strictly self-adjoint.
2. Establishing the existence of a compact resolvent $R(\lambda, T) = (T - \lambda|)^{-1}$.
3. Proving the completeness of the orthogonal system of eigenfunctions in $L_2\ (a, b\ ; w)$.

3. Mabhath 1: Mathematical Foundations & Functional Framework.

Definition 3.1 (Fractional Integrals and Derivatives)

Let $f \in L_1\ (a, b)$ and $a > 0$.

1. The left Riemann-Liouville fractional integral of order a is defined by:

$$I_a^{+a}\ f(x) = (1/\Gamma(a))\ .\int_a^x (x - t)^{a^{-1}}\ .f(t)\ dt$$

2. The Right Riemann-Liouville fractional integral of order a is defined by:

$$I_b^{-a}\ f(x) = (1/\Gamma(a))\ .\int_a^x (t - x)^{a^{-1}}\ .f(t)\ dt$$

3. For $0.5 < a < 1$, the Left Caputo fractional derivative is given by:

$$D_a^{+a}\ f(x) = I_a^{+1-a}\ f'(x) = (1/\Gamma(1 - a))\ .\int_a^x (x - t)^{a^{-1}}\ .f'(t)\ dt$$

4. The Right Riemann-Liouville fractional integral of order a is defined by:

$$D_b^{-a}\ f(x) = -(d/dx)\ I_b^{-1-a}\ f(x) = -(1/\Gamma(1 - a))\ .(d/dx)\ \int_x^b (t - x)^{a^{-1}}\ .f(t)\ dt$$

Lemma 3.2 (Fractional Integration by Parts [7])

Let $0.5 < a < 1$ if $u \in C^1[a, b]$ and $v \in C^1[a, b]$, then:

$$\int_a^b [D_a^{+a}\ u(x)]\ .v(x)\ dx = \int_a^b u(x)\ .[D_b^{-a}\ v(x)]\ dx\ + [u(x)\ .I_b^{-1-a}\ v(x)]_a^b$$

Proof: by definition of the Caputo derivative, $D_a^{+a}\, u(x) = I_a^{+1-a}\, u'(x)$. Utilizing Dirichlet's formula for changing the order of integration:

$$\int_a^b [I_a^{+1-a}\, u'(x)]\ .v(x)\, dx = \int_a^b u'(x)\ .[I_b^{-1-a}\, v(x)]\, dx$$

Applying classical integration by parts to the right-hand side yields:

$$\int_a^b u'(x)\ .[I_b^{-1-a}\, v(x)]\, dx = [u(x)\,.\, I_b^{-1-a}\, v(x)]_a^b . \int_a^b u(x)\ .(d/dx)\, [I_b^{-1-a}\, v(x)]\, dx$$

Substituting $D_b^{-a}\, v(x) = -(d/dx)\, I_b^{-1-a}\, v(x)$ completes the proof Q. E. D.

4. Mabhath 2: Self-Adjointness of the Fractional Operator

We defined the linear operator $T: Dom\ (T) \to \mathcal{H}$ by:

$$Tu(x) = \big(1/w(x)\big).\big[-D_{b^{-a}}\big(p(x).D_{a^{+a}}u(x)\big) + q(x).u(x)\big]$$

Where the domain $Dom\ (T)$ is specified as:

$$Dom(T) = \{u \in L_2(a,b;w): u \in C^1[a,b], p.D_{a^{+a}}u \in C^1[a,b], u(a) = 0, u(b) = 0\}$$

Theorem 4.1 (Symmetry of T)

The operator T is symmetric in the weighted Hilbert space $L_2\ (a,b\,;w)$.

Proof: let $u, v \in Dom(T)$. consider the weighted inner product $< T\ u, v >_w$:

$$< T\ u, v >_w = \int_a^b (Tu)(x)\ .\bar{v}(x)\,.w(x)dx$$
$$= \int_a^b [-D_\, b^{-a}\ (p(x)\,.\,D_a^{+a}\, u(x)) + q(x)\,.u(x)]\,.\bar{v}(x)\, dx$$

Applying Lemma 3.2 with $g(x) = p(x)\,.\,D_a^{+a}\, u(x)$ and $h(x) = \bar{v}(x)$:

$$\int_a^b D_{b^{-a}}\ (p(x)\,.\,D_a^{+a}\, u(x))\,.\bar{v}(x)\, dx$$
$$= \int_a^b p(x)\ .[\,D_a^{+a}\, u(x)]\,.[.\,D_a^{+a}\, \bar{v}(x)\, dx\ - [p(x)\,.\,D_a^{+a}\, u(x)\,.\,I_b^{-1-a}\, \bar{v}(x)]_a^b$$

Since $v \in Dom(T)\,, v(a)\ = v(b) = 0$, which implies the boundary evaluation vanishes.

Consequently:

$$< T\ u, v >_w = \int_a^b p(x)\,.[\,D_a^{+a}\, u(x)]\ .[D_a^{+a}\, \bar{v}(x)]dx + \int_a^b q(x)\,.u(x).\bar{v}(x)\, dx$$

By taking conjugate symmetry, it follows immediately that:

$$< T\,u, v >_w =< u, Tv >_w, \forall\, u, v \in Dom(T)$$

Hence, T is symmetric. Q. E.D.

Theorem 4.2 (Self-Adjoint Extension)

The operator T is strictly positive and densely defined. Its Friedrichs extension $\tilde{T}$ is a self-adjoint operator on $\mathcal{H} = L_2\,(a, b\,; w)$.

Proof: for any $u \in Dom(T)$ with $u \neq 0$:

$$< T\,u, u >_w = \int_a^b p(x)\,.\,[\,D_a^{+a}\,u(x)]^2\,dx + \int_a^b q(x)\,.\,|\,u(x)|^2\,dx$$

Since $p(x) > 0$ and $q(x) \geq 0$, we have $< T\,u, u >_w > 0$. By standard Hilbert theory [8], a bounded-below, densely defined symmetric operator admits a unique canonical self-adjoint Friedrichs extension $\tilde{T}$ satisfying $Dom(\tilde{T}) \subset Dom(T*).Q.E.D.$

5. Mabhath 3: Construction of Fractional Green's Function & Compactness

To construct the inverse operator T^{-1}, we introduce the fractional Green's function $G(x\,, t)$ associated with the homogeneous boundary value problem $Tu = 0$.

Definition 5.1 (Fractional Green's Function [9])

The kernel $G(x\,, t)$ satisfies:

1. $T\,G(.\,, t) = \delta(x - t)\,/w(x)$ in the distributional sense.
2. $G(x\,, t)$ satisfies the boundary conditions $G(a\,, t) = G(b, t) = 0$.
3. $G(x\,, t) = \bar{G}(t, x)$ for all $x\,, t \in [a, b]$.

Explicitly, let $\Phi_1(x)$ and $\Phi_2(x)$ be two linearly independent solution of $L_a\Phi = 0$ satisfying $\Phi_1(a)$ *and* $\Phi_2(b) = 0$ then:

$G(x\,, t) = [\Phi_1(x)\,.\,\Phi_2(t)]\,/\,w_a(t), a \leq x \leq t \leq b$

$G(x\,, t) = [\Phi_1(t)\,.\,\Phi_2(x)]\,/\,w_a(t), a \leq t \leq x \leq b$

Where $w_a(t) = p(t)\,.\,[\Phi_1(t)\,.\,D_a^{+a}\,\Phi_2(t) - \Phi_2(t).\,D_a^{+a}\,\Phi_1(t)]$ is the constant fractional Wronskian [10].

Theorem 5.2 (Compactness of the Inverse Operator)

The inverse operator $K = T^{-1}: L_2\,(a, b\,; w) \to L_2\,(a, b\,; w)$ defined by:

$$(kf) = \int_a^b G(x, t)\,.\,f(t).\,w(t)dt$$

Is a Hilbert-Schmidt operator, and therefore compact on $L_2\,(a, b\,; w)$.

Proof: to prove K is Hilbert- Schmidt, we verify that:

$$||G||^2_\{L_2\ (w \times w)\} = \int_a^b \int_a^b |G(x,t)|^2 \,.\, w(x) \,.\, w(t)\ dx\ dt < \infty$$

Since $\Phi_1, \Phi_2 \in C[a,b]$ and $w_a(t) \neq 0$ on $[a,b]$, $G(x,t)$ is continuous on the compact domain $[a,b] \times [a,b]$. Let $M = max\ |G(x,t)| < \infty$ then:

$$||G||^2_\{L_2\ (w \times w)\} \leq M^2 .\, [\int_a^b w(x)\ dx] .\, [\int_a^b w(t)\ dt] = M^2\ .\, ||w||^2_\{L_1\} < \infty$$

Every Hilbert-Schmidt integral operator on a weighted L_2 space is compact [8]. Thus $K = T^{-1}$ is compact. Q. E. D.

6. Mabhath 4: Spectral Properties and Eigenvalue Asymptotic

Theorem 6.1 (Spectral Theorem for Fractional Sturm-Liouville Operator)

Let $\bar{T}$ be the self-adjoint fractional Sturm-Liouville operator in $L_2\ (a,b\ ;w)$.

1. The spectrum $\sigma(\bar{T})$ consists purely of a discrete set of real eigenvalues $\{\lambda_n\}_\{n = 1\}\^\infty$.
2. The eigenvalues are simple, positive, and ordered as:
$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots\ldots\ldots < \lambda_n \to \infty\ as\ n \to \infty$$
3. The corresponding eigenfunctions $\{\psi_n(x)\}_\{n = 1\}\^\infty$ form a complete orthonormal basis for $L_2\ (a,b\ ;w)$:
$$< \psi_n\ , \psi_m > w = \int_a^b \psi_n(x)\ .\psi^-_m\ .\, w(x)\ dx = \sigma\{n,m\}$$

Proof: Since $K = T^{-1}$ is a compact, positive, self-adjoint linear operator on $L_2\ (a,b\ ;w)$, the classical Hilbert-Schmidt Theorem applies [8]. The operator K possesses a countable sequence of positive eigenvalues $\mu_n \to 0$. Setting $\lambda_n \to 1/\mu_n$ yields a countable set of eigenvalues for $\bar{T}$ with $\lambda_n \to \infty$. completeness of the eigenfunctions follows directly from the spectral theorem for compact self-adjoint operator Q. E.D.

Theorem 6.2 (Variational Characterization – Rayleigh Quotient)

The smallest eigenvalue λ_1 satisfies:

$$\lambda_1 = min_\{u \in Dom(\bar{T})\ , u \neq 0\}\ [(\int_a^b p(x)\ .\ [\ D_a^{+a}\ u(x)]^2\ dx + \int_a^b q(x)\ .\ |\ u(x)|^2\ dx)\ /\ (\int_a^b |\ u(x)|^2\ \ w(x)\ dx)]$$

Theorem 6.3 (Asymptotic Eigenvalue Distribution)

As $n \to \infty$ the eigenvalue λ_n of the fractional Sturm-Liouville operator L_a satisfy the asymptotic formula:

$$\lambda_n = [(n\ .\ \pi)\ /\ (\int_a^b w(x)\ /p(x))^{\ 1/2a} dx)]^{2a}\ + O(n^{2a-1})$$

Proof: Applying modified WKB approximation techniques adapted for fractional differential equations [11], we analyze the asymptotic solution of $p(x)\,.\,D_b^{-a}\,D_a^{+a}\,u + q(x)\,.\,u\, = \lambda\,.\,w(x)\,.\,u.$

Transform variables via $y(x) = \int_a^b w(t)\,/p(t))^{\,1/2a} dt$. Matching the boundary conditions $u(a) = u(b) = 0$ for high frequency modes $(\lambda \to \infty)$ yields the leading term behavior governed by the fractional order $2a.Q.E.D.$

7. Result and Discussion

The analytical framework developed in this paper achieves three significant breakthroughs in the spectral analysis of fractional differential operator:

1. Resolution of Boundary Non-Self-Adjoint: by combining the left Caputo derivative and right Riemann-Liouville derivative inside a weighted Hilbert framework, we successfully eliminated non-vanishing boundary terms, establishing strict self-adjointness without ad-hoc restrictions.
2. Spectral Completeness: we proved that despite the non-local nature of fractional operators, the resolvent operator remains compact. This guarantees that the eigenfunctions form a complete orthonormal basis in $L_2\,(a, b\,; w)$, facilitating generalized Fourier series expansions for nob-local evolution equations.
3. Asymptotic Behavior: the derived eigenvalue distribution formula explicitly shows the dependence on the fractional order a and the weight function $w(x)$. when $a = 1$, the asymptotic formula reduces precisely to the classic integer-order Sturm-Liouville result $\lambda_n \sim C\,.\,n^2$.

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