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Analytical Framework of the ZR-Transform: Duality Relations, Operational Properties, and Exact Solutions of ODEs

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ABSTRACT

We present a mathematical study of the *ZR-transform*, focusing on its foundational properties and operational calculus over the semi-infinite interval $[0,\infty)$. We derive its key theoretical aspects, including existence and uniqueness theorems under exponential order conditions, alongside standard operational rules for linearity, shifting, scaling, and convolution. To examine its algebraic structure, we establish direct duality relations between the *ZR-Trans*form and classical tools like the Laplace transform. Furthermore, exact operational formulas are developed for higher-order derivatives and polynomial-multiplied functions. To evaluate the transform's practical performance, we apply it to solve initial value problems governed by linear ordinary differential equations, showing that it provides an exact and straightforward alternative for differential systems in applied mathematics and physics.

1. Introduction

Integral transforms are standard tools for solving linear differential, integro-differential, and dynamic equations by mapping complex operators into tractable algebraic systems [1, 2]. Following the foundational work on classical Fourier and Laplace transforms [2, 3], several specialized integral operators including the Sumudu [4], Elzaki [5], Sadik [6], and Aboodh [7] transforms have been developed to address specific boundary value problems, singular kernels, and fractional systems [3, 8]. These modern operators often simplify scaling behavior and preserve physical dimensions during solving procedures [4, 9].

To address these considerations, we introduce the theoretical framework of the *ZR-Transform* for integrable functions defined on $[0,\infty)$. The proposed transform incorporates a dual-scale kernel involving s^2 and a logarithmic weight $\ln(s+1)$, which sets it apart from existing integral operators like the Laplace, Sumudu, and Elzaki transforms. This architecture provides an added logarithmic dampening factor in the transform domain, facilitating convergence for functions with rapid growth. Table 1 summarizes the structural differences between the *ZR-Transform* and classical integral operators.

Table 1: Comparison of the proposed *ZR-Transform* with classical integral transforms.

Transform	**Kernel / Operational Definition**	**Reference**
Laplace Transform	$L\left[f(t)\right]=\int_0^{\infty} f(t)e^{-st}dt$	[3]
Sumudu Transform	$G(u)=\frac{1}{u}\int_0^{\infty} f(t)e^{-\frac{t}{u}}dt$	[4]

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Transform	Kernel / Operational Definition	Reference
Elzaki Transform	$E(v)=v\int_0^{\infty} f(t)e^{-\frac{t}{v}}dt$	[5]
Proposed ZR-Transform	$\mathcal{ZR}(s)=ZR[f(t)]=\frac{1}{s\ln(s+1)}\int_0^{\infty} f(t)\,e^{-s^2t}dt$	Present Work

2. Preliminaries

In this section, we review the fundamental concepts, function spaces, classical integral transforms, and established analytical theorems required to construct the theoretical framework of the proposed ZR-Transform.

Definition 2.1 (function spaces and Exponential order)

Let $PC[0,\infty)$ denote the class of piecewise continuous function defined on the semi-infinite interval $[0,\infty)$. A function $f(t)\in PC[0,\infty)$ is said to be of exponential order $e^{\mu t}$ as $t\to\infty$ if there exist positive constants M and T such that:

$|f(t)|\le Me^{\mu t}\ ,\forall t\ge T$

Where $\mu>0$. The exponential order condition guarantees the absolute convergence of infinite integral operators with exponential kernels [1,2].

Theorem 2.2 (Lerch's Theorem on Uniqueness)

If $f(t)$ and $g(t)$ are piecewise continuous function on $[0,\infty)$ of exponential order such that their integral transform representations coincide, i.e., $F(s)=G(s)$ for all $s>\mu$, then $f(t)=g(t)$ at all points where both functions are continuous [10].

Theorem 2.3 (Leibniz Integral Rule)

Let $f(t,s)$ and its partial derivative $\frac{\partial f}{\partial s}$ be continuous functions in t and s over a given region the differentiation under the integral sign is governed by:

$$\frac{d}{ds}\int_{a(s)}^{b(s)} f(t,s)\,dt=f(b(s),s)\cdot b'(s)-f(a(s),s)\cdot a'(s)+\int_{a(s)}^{b(s)}\frac{\partial f(t,s)}{\partial s}dt\ .$$

For infinite, uniform convergence of the integral validates the interchange of differentiation and integration operators [1].

3. Definition and Basic properties of ZR-Transform

3.1 Definition (ZR-transform)

Let $f(t)$ be a function that is integrable on $[0,\infty)$. The *ZR-Transform* of $f(t)$, denoted by $\mathcal{ZR}(s)=ZR[f(t)]$, is defined for $s>0$ by the integral formula:

$$\mathcal{ZR}(s)=ZR[f(t)]=\frac{1}{s\ln(s+1)}\int_0^{\infty} f(t)\,e^{-s^2t}dt \qquad (1)$$

3.2 Proposition (Linearity property)

Let $f_1(t), f_2(t), ..., f_n(t)$ be integrable functions for $ZR\left[f_1(t)\right], ZR\left[f_2(t)\right], ..., ZR\left[f_n(t)\right]$, respectively, then the linearity property is defined by:

$ZR[b_1 f_1(t) + b_2 f_2(t) + ... + b_m f_n(t)] = b_1 ZR[f_1(t)] + b_2 ZR[f_2(t)] + ... + b_m ZR[f_n(t)]$ for any constants $b_1, b_2, ..., b_m$

Proof

Since definition of *ZR-transform* is $ZR[f\ (t)] = \dfrac{1}{s\ln(s+1)} \int_0^\infty f\ (t) e^{-s^2 t} dt$, then, we get:

$$ZR[b_1 f_1(t) + b_2 f_2(t) + ... + b_m f_n(t)] = \frac{1}{s\ln(s+1)} \int_0^\infty [b_1 f_1(t) + b_2 f_2(t) + ... + b_m f_n(t)] e^{-s^2 t} dt,$$

$$= \frac{1}{s\ln(s+1)} \int_0^\infty b_1 f_1(t) e^{-s^2 t} dt + \frac{1}{s\ln(s+1)} \int_0^\infty b_2 f_2(t) e^{-s^2 t} dt + ... + \frac{1}{s\ln(s+1)} \int_0^\infty b_m f_n(t) e^{-s^2 t} dt,$$

$$= \frac{b_1}{s\ln(s+1)} \int_0^\infty f_1(t) e^{-s^2 t} dt + \frac{b_2}{s\ln(s+1)} \int_0^\infty f_2(t) e^{-s^2 t} dt + ... + \frac{b_m}{s\ln(s+1)} \int_0^\infty f_n(t) e^{-s^2 t} dt,$$

$$= b_1 ZR[f_1(t)] + b_2 ZR[f_2(t)] + ... + b_m ZR[f_n(t)].$$

3.3 Theorem (Linearity of the Inverse Transform)

Let $ZR[f\ (t)] = \mathcal{ZR}_1(s)$ and $ZR[g(t)] = \mathcal{ZR}_2(s)$. For any constant $a, b \in R$:

$$ZR^{-1}\left\{a\mathcal{ZR}_1(s) \pm b\,\mathcal{ZR}_2(s)\right\} = aZR^{-1}\{\mathcal{ZR}_1\} \pm b\{\mathcal{ZR}_2\}.$$

Proof

Since the transform *ZR-transform* is linear, then:

$$ZR\left\{af\ (t) \pm bg(t)\right\} = aZR\{f\ (t)\} \pm bZR\{g(t)\} = \text{a}\ \mathcal{ZR}_1 \pm b\,\mathcal{ZR}_2.$$

Applying the inverse transform ZR^{-1} to both sides yields:

$$ZR^{-1}\left\{a\mathcal{ZR}_1 \pm b\,\mathcal{ZR}_2\right\} = af\ (t) \pm bg(t) = \text{a}\ ZR^{-1}\left\{\mathcal{ZR}_1\right\} \pm bZR^{-1}\left\{\mathcal{ZR}_2\right\}.$$

Thus, ZR^{-1} is a linear operator.

3.4 Theorem: (Existence and Uniqueness)

If a function $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order $e^{\mu t}$ for $\mu > 0$, (i.e. $(\left|f\ (t)\right| \le Ae^{\mu t}$ for some $A > 0)$ then the *ZR-Transform* of $f(t)$ exists and is unique for all $s > \mu^2$.

Proof

1. Existence

To prove that the *ZR-Transform* exists, for a positive constant A, we have:

$$\left|ZR\{f(t)\}\right| = \left|\frac{1}{s\ln(s+1)}\int_0^\infty f(t)e^{-s^2t}dt\right| \le \frac{1}{s\ln(s+1)}\int_0^\infty |f(t)|\,e^{-s^2t}dt \le \frac{1}{s\ln(s+1)}\int_0^\infty Ae^{-\mu t}e^{-s^2t}dt$$

$$\le \frac{A}{s\ln(s+1)(\mu - s^2)} < \infty, \quad \text{for } s > \mu^2$$

Thus, the integral converges and $ZR\{f(t)\}$ exists.

2. Uniqueness

Assume that $f(t)$ and $g(t)$ be two continuous functions such that:

$ZR\{f(t)\} = ZR\{g(t)\}$. By linearity, $ZR\{f(t) - g(t)\} = 0$. By Lerch's theorem this implies $f(t) - g(t) = 0$ at all points of continuity. Hence, $f(t) = g(t)$, establishing uniqueness.

Theorem 3.4.1 (Logarithmic Rate of Convergence Acceleration)

Let $f(t) \in PC[0,\infty)$ be of exponential order $e^{\mu t}$ with $\mu > 0$. Then for all $s^2 > \mu$, the transform satisfies the following decay relation relative to the Laplace transform:

$$\lim_{s\to\infty} \frac{\left|ZR[f(t)]\right|}{\left|L[f(t);s^2]\right|} = 0$$

Proof:

By combining the integral definitions of the *ZR-Transform* and the Laplace transform, we obtain:

$$ZR[f(t)] = \frac{1}{s\ln(s+1)} L[f(t);s^2]$$

Taking absolute values on both sides gives:

$$\left|ZR[f(t)]\right| = \frac{1}{s\ln(s+1)}\left|L[f(t);s^2]\right|$$

Taking the limit as $s \to \infty$

$$\lim_{s\to\infty}\left|ZR[f(t)]\right| = \lim_{s\to\infty}\frac{1}{s\ln(s+1)}\left|L[f(t);s^2]\right| = 0$$

This demonstrates that the factor $\frac{1}{s\ln(s+1)}$ introduces a logarithmic dampening effect, ensuring a faster rate of attenuation in the transform domain compared to standard exponential scaling.

3.5 ZR-transform for some elementary functions

Assume that for any function $f(t)$, the integral in equation (1) exists. So *ZR-transform* will be:

Function $f(t) = ZR^{-1}\{\mathcal{ZR}(s)\}$	$\mathcal{ZR}(s) = ZR[f(t)]$
$f(t) = k$	$ZR[f(t)] = \frac{k}{s^3\ln(s+1)}$

$f(t)=t$	$ZR[f(t)]=\frac{1}{s^5\ln(s+1)}$
$f(t)=t^n$	$ZR[f(t)]=\frac{n!}{s^{2n+3}\ln(s+1)}$
$f(t)=\sin at$	$ZR[f(t)]=\frac{a}{(s^4+a^2)s\ln(s+1)}$
$f(t)=\cos at$	$ZR[f(t)]=\frac{s^2}{(s^4+a^2)s\ln(s+1)}$
$f(t)=\sinh at$	$ZR[f(t)]=\frac{a}{(s^4-a^2)s\ln(s+1)}$
$f(t)=\cosh at$	$ZR[f(t)]=\frac{s^2}{(s^4-a^2)s\ln(s+1)}$
$f(t)=e^{at}$	$ZR[f(t)]=\frac{1}{(s^2-a)s\ln(s+1)}$

3.6 Theorem (Derivatives of ZR-Transform)

Assume that $f(t)$ be n-terms continuously differentiable on $[0,\infty)$. The *ZR-Transform* of the derivative $f'(t)$ and higher-order derivatives $f^{(n)}(t)$ are given by:

1. $ZR[f'(t)]=s^2ZR[f(t)]-\frac{1}{s\ln(s+1)}f(0)$,
2. $ZR[f''(t)]=s^4ZR[f(t)]-\frac{s}{\ln(s+1)}f(0)-\frac{1}{s\ln(s+1)}f'(0)$,
3. $ZR[f^{(n)}(t)]=(s^2)^nZR[f(t)]-\frac{1}{s\ln(s+1)}\sum_{k=0}^{n-1}(s^2)^{n-1-k}f^{(k)}(0)$.

Proof

1. Since $\mathcal{ZR}(s)=ZR[f(t)]=\frac{1}{s\ln(s+1)}\int_0^\infty f(t)e^{-s^2t}dt$, integrating by parts:

Then: $ZR[f'(t)]=s^2ZR[f(t)]-\frac{1}{s\ln(s+1)}f(0)$.

2. since $\mathcal{ZR}(s)=ZR[f(t)]=\frac{1}{s\ln(s+1)}\int_0^\infty f(t)e^{-s^2t}dt$, integrating by parts twice:

$ZR[f''(t)]=s^4ZR[f(t)]-\frac{s}{\ln(s+1)}f(0)-\frac{1}{s\ln(s+1)}f'(0)$

3. For this case (n^{th} derivative), it can be proved using mathematical induction.

3.7 Theorem (convolution of ZR-transform)

Let $f_1(t)$ and $f_2(t)$ be integrable functions with *ZR-transform* $\mathcal{ZR}_1(s)$ *and* $\mathcal{ZR}_2(s)$ respectively, i.e. $ZR\{f_1(t)\} = \mathcal{ZR}_1(s)$ *and* $ZR\{f_2(t)\} = \mathcal{ZR}_2(s)$ .then it can be shown that:

$ZR\{(f_1 * f_2)(t)\} = s\ln(s+1)\mathcal{ZR}_1(s)\mathcal{ZR}_2(s)$

Proof

By the definition of the *ZR-Transform*:

$$ZR(f_1 * f_2) = \frac{1}{s\ln(s+1)}\int_0^\infty e^{-s^2 t}\left[\int_0^t f_1(\tau) f_2(t-\tau)d\tau\right]dt,$$

Expressing the right-hand side as a double integral over the integration region $D = \{(t,\tau): 0 \le \tau \le t < \infty\}$:

$$I = \int_0^\infty \int_0^t e^{-s^2 t} f_1(\tau) f_2(t-\tau)d\tau dt\ .$$

Since $f_1(t)$ and $f_2(t)$ are piecewise continuous and of exponential order, the integral converges absolutely for $s^2 > \mu$ Applying Fubini's theorem allows us to interchange the order of integration:

$$I = \int_0^\infty f_1(\tau)\left(\int_\tau^\infty e^{-s^2 t} f_2(t-\tau)dt\right)d\tau$$

Substituting $u = t-\tau$ wicht yields $= u+\tau$ and $dt = du$ with $u \in [0,\infty)$:

$$I = \int_0^\infty f_1(\tau)e^{-s^2\tau}\left(\int_0^\infty e^{-s^2 u} f_2(u)du\right)d\tau = \left(\int_0^\infty f_1(\tau)e^{-s^2\tau}d\tau\right)\left(\int_0^\infty f_2(u)e^{-s^2 u}du\right).$$

Multiplying and dividing the right-hand side by $[s\ln(s+1)]^2$:

$$I = \left[s\ln(s+1)\,\mathrm{ZR}\left[f_1(t)\right]\right].\left[s\ln(s+1)\,\mathrm{ZR}\left[f_2(t)\right]\right]$$

Substituting this result back into the original transform equation gives:

$$ZR(f_1 * f_2) = \frac{1}{s\ln(s+1)}.s^2\ln^2(s+1)\mathcal{ZR}_1(s)\mathcal{ZR}_2(s)$$

$$= s\ln(s+1)\mathcal{ZR}_1(s)\mathcal{ZR}_2(s)$$

3.8 Theorem (duality Relation between ZR-Laplace Transform)

If $\mathcal{ZR}(s)$ is *ZR-transform* and $F(p)$ is Laplace transform for any integrable function $f(t)$ on $[0,\infty)$, then: $\mathcal{ZR}(s) = \frac{1}{s\ln(s+1)}F(s^2)$

Proof

Since $\mathcal{ZR}(s) = ZR[f(t)] = \frac{1}{s\ln(s+1)}\int_0^\infty f(t)e^{-s^2 t}dt$.

And Laplace transform is denoted by:

$F(p)=\mathcal{L}[f(t);p]=\int_0^{\infty}E^{-pt}f(t)dt$, then the duality will be:

$$\mathcal{ZR}(s)=\frac{1}{s\ln(s+1)}F(s^2).$$

3.9 Proposition (scaling property)

Suppose that $f(t)$ be an integrable function on $[0,\infty)$ and let $ZR[f(t)]=\mathcal{ZR}(s)$ For any constant $\lambda>0$, then: $ZR[f(\lambda t)]=\frac{\ln(\frac{s}{\sqrt{\lambda}}+1)}{\lambda\sqrt{\lambda}\ln(s+1)}\mathcal{ZR}(\frac{s}{\sqrt{\lambda}})$.

Proof

Since $\mathcal{ZR}(s)=ZR[f(t)]=\frac{1}{s\ln(s+1)}\int_0^{\infty}f(t)e^{-s^2t}dt$,then: $ZR[f(\lambda t)]=\frac{1}{s\ln(s+1)}\int_0^{\infty}f(\lambda t)e^{-s^2t}dt$

$Let\ \lambda t=x\rightarrow t=\frac{x}{\lambda}\rightarrow dt=\frac{dx}{\lambda}$,

Substitute assumption in last equation, then:

$$ZR[f(\lambda t)]=\frac{1}{\lambda s\ln(s+1)}\int_0^{\infty}f(x)e^{-(\frac{s}{\sqrt{\lambda}})^2x}dx,$$

$$ZR[f(\lambda t)]=\frac{\ln(\frac{s}{\sqrt{\lambda}}+1)}{\lambda\sqrt{\lambda}\ln(s+1)}\mathcal{ZR}(\frac{s}{\sqrt{\lambda}}).$$

3.10 Proposition (first shifting property)

If ZR-transform of the function $f(t)$ is $\mathcal{ZR}(s)$ then ZR-transform of function $e^{at}f(t)$,(a is any real constant) is given by:

$$ZR[e^{at}f(t)]=\frac{\sqrt{s^2-a}\ln(\sqrt{s^2-a}+1)}{s\ln(s+1)}\mathcal{ZR}(\sqrt{s^2-a})$$

Proof

Since $\mathcal{ZR}(s)=ZR[f(t)]=\frac{1}{s\ln(s+1)}\int_0^{\infty}f(t)e^{-s^2t}dt$

$$ZR[e^{at}f(t)]=\frac{1}{s\ln(s+1)}\int_0^{\infty}e^{at}f(t)e^{-s^2t}dt,$$

$$=\frac{1}{s\ln(s+1)}\int_0^{\infty}f(t)e^{-(s^2-a)t}dt,$$

Then, we get:

$$\int_0^{\infty}f(t)e^{-(s^2-a)t}dt=\sqrt{s^2-a}\ln(\sqrt{s^2-a}+1)\mathcal{ZR}(\sqrt{s^2-a})$$

$$ZR[e^{at}f(t)]=\frac{\sqrt{s^2-a}\ln(\sqrt{s^2-a}+1)}{s\ln(s+1)}\mathcal{ZR}(\sqrt{s^2-a})$$

3.11 Proposition (second shifting property)

Assume that $f(t)$ be an integrable function and

$ZR[H(t-a)f(t-a)] = \frac{1}{s\ln(s+1)}\int_0^\infty [H(t-a)f(t-a)]e^{-s^2 t}dt$, (a is any real constant).

$ZR[H(t-a)f(t-a)] = e^{-as^2} ZR[f(t)]$.

Proof

Since from equation (2.1), we have: $\mathcal{ZR}(s) = ZR[f(t)] = \frac{1}{s\ln(s+1)}\int_0^\infty f(t)e^{-s^2 t}dt$,then:

$ZR[H(t-a)f(t-a)] = \frac{1}{s\ln(s+1)}\int_0^\infty [H(t-a)f(t-a)]e^{-s^2 t}dt$,

Where:

$H(t-a) = \begin{cases} 0 & if \ \ 0 \le t \le a \\ 1 & if \ \ t > a \end{cases}$,

then:

$$ZR[H(t-a)f(t-a)] = \frac{1}{s\ln(s+1)}\left[\int_0^a 0.f(t-a)e^{-s^2 t}dt + \int_a^\infty 1.f(t-a)e^{-s^2 t}dt\right]$$

$$= \frac{1}{s\ln(s+1)}\int_a^\infty f(t-a)e^{-s^2 t}dt.$$

let $x = t-a \to t = x+a \to dt = dx \ \ and \ \ t = a \to x = 0, t = \infty \to x = \infty$

By use last assumption, we get:

$$ZR[H(t-a)f(t-a)] = \frac{1}{s\ln(s+1)}\int_0^\infty f(x)e^{-s^2(x+a)}dx,$$

$$= \frac{e^{-as^2}}{s\ln(s+1)}\int_0^\infty f(x)e^{-s^2 x}dx,$$

Then:

$ZR[H(t-a)f(t-a)] = e^{-as^2} ZR[f(t)]$.

3.12 Theorem :(Derivative of the ZR-Transform n-order)

Let $f(t)$ be an integrable function on $[0,\infty)$ such that $\frac{1}{s\ln(s+1)}, s^2$ are differentiable functions and $\left(s^2\right)' \neq 0$ then:

1. $ZR[tf(t)] = \frac{1}{s\ln(s+1)}\left[\frac{-1}{2s}\frac{d}{ds}(s\ln(s+1)ZR[f(t)])\right]$,
2. $ZR[t^2 f(t)] = \frac{1}{s\ln(s+1)}\left[\left(\frac{-1}{2s}\frac{d}{ds}\right)^2 s\ln(s+1)ZR[f(t)]\right]$,
3. $ZR[t^n f(t)] = \frac{1}{s\ln(s+1)}(\frac{-1}{2s}\frac{d}{ds})^n (s\ln(s+1)ZR[f(t)]), \ \ n \in N$.

Proof

1. since $ZR[f(t)] = \frac{1}{s\ln(s+1)}\int_0^\infty f(t)e^{-s^2t}dt$,

   We have: $s\ln(s+1)ZR[f(t)] = \int_0^\infty f(t)e^{-s^2t}dt$.

   Differentiating both sides with respect to s, we get:

   $\frac{d}{ds}\left[s\ln(s+1)ZR[f(t)]\right] = \frac{d}{ds}\left[\int_0^\infty f(t)e^{-s^2t}dt\right] \rightarrow = \int_0^\infty f(t)(-2st)e^{-s^2t}dt$,

   Then:

   $\frac{d}{ds}\left[s\ln(s+1)ZR[f(t)]\right] = -2s\int_0^\infty tf(t)e^{-s^2t}dt$.

   Rearranging terms:

   $\int_0^\infty tf(t)e^{-s^2t}dt = \frac{-1}{2s}\frac{d}{ds}\left[s\ln(s+1)ZR[f(t)]\right]$.

Multiplying both sides by $\frac{1}{s\ln(s+1)}$ yields:

$$ZR[tf(t)] = \frac{1}{s\ln(s+1)}\left[\frac{-1}{2s}\frac{d}{ds}(s\ln(s+1)ZR[f(t)])\right].$$

2. For $n=2$, differentiating again with respect to s gives the second order derivative operator $\left(\frac{-1}{2s}\frac{d}{ds}\right)^2$.

from the first part, we have: $\int_0^\infty tf(t)e^{-s^2t}dt = (\frac{-1}{2s}\frac{d}{ds})\left[s\ln(s+1)ZR[f(t)]\right]$

Differentiating both sides again with respect to s:

$$s\ln(s+1)ZR[t^2f(t)] = \left(\frac{-1}{2s}\frac{d}{ds}\right)^2\left[s\ln(s+1)ZR[f(t)]\right].$$

Then:

$$ZR[t^2f(t)] = \frac{1}{s\ln(s+1)}\left[\left(\frac{-1}{2s}\frac{d}{ds}\right)^2 s\ln(s+1)ZR[f(t)]\right]$$

3. Generalization for t^n follows directly using the principle of mathematical induction, we get:

$$ZR[t^nf(t)] = \frac{1}{s\ln(s+1)}(\frac{-1}{2s}\frac{d}{ds})^n(s\ln(s+1)ZR[f(t)]).$$

3.13 Periodic of ZR-Transform

Periodic function is a sectionally continuous function which means that for some $\lambda > 0$ satisfies:
$f(t) = f(t + n\lambda)$

We can write *ZR-transform* of $f(t)$ as a series of integrals as follows:

$$ZR[f(t)] = \frac{1}{s\ln(s+1)} \sum_{n=0}^{\infty} \int_{0}^{(n+1)\lambda} f(t) e^{-s^2 t} dt,$$

Let $t = x + n\lambda$, we get:

$$ZR[f(t)] = \frac{1}{s\ln(s+1)} \left(\sum_{n=0}^{\infty} e^{-ns^2\lambda}\right) \int_{0}^{\lambda} f(x) e^{-s^2 x} dx,$$

Since the summation of geometric series is $\sum_{n=0}^{\infty} \left(e^{-s^2\lambda}\right)^n = \frac{1}{1-e^{-s^2\lambda}}$.

$$ZR[f(t)] = \frac{1}{s\ln(s+1)(1-e^{-s^2\lambda})} \int_{0}^{\lambda} f(t) e^{-s^2 t} dt.$$

Illustrative Examples

The following examples explain how *ZR-transform* solves initial value problems that are described by ordinary differential equations

Example 1

Consider the following ordinary differential equation:

$$\begin{cases} x'(t) - y(t) = 0 \\ y'(t) + x(t) = 0 \end{cases}, \qquad x(0) = 1, y(0) = 0$$

Solution:

Applying *ZR-transform* for both sides of the original equation to get:

$$ZR[x'] - ZR[y] = 0 \ldots.(1)$$

$$ZR[y'] + ZR[x] = 0 \ldots.(2)$$

$$s^2 ZR[x] - \frac{1}{s\ln(s+1)} x(0) - ZR[y] = 0 \ldots.(1)$$

$$s^2 ZR[y] - \frac{1}{s\ln(s+1)} y(0) + ZR[x] = 0 \ldots.(2)$$

Using the initial conditions, to obtain:

$$s^2 ZR[x] - \frac{1}{s\ln(s+1)} - ZR[y] = 0 \ldots.(1)$$

$$s^2 ZR[y] + ZR[x] = 0 \ldots.(2)$$

From equation (2), we obtain

$$ZR[x] = -s^2 ZR[y] \ldots.(3)$$

Substituting equation (3) into equation (1), we get:

$$ZR[y] = \frac{-1}{(s^4+1)s\ln(s+1)}$$

Taking the inverse of *ZR-transform* to get

$y = -\sin t$

From equation (3), we get

$$ZR[x] = \frac{s^2}{(s^4+1)s\ln(s+1)}$$

Taking the inverse of *ZR-transform* to get

$x = \cos t$.

Example 2

Consider the following second-order initial value problem:

$y''(t) + 3y'(t) + 2y(t) = 0, \quad y(0) = 1, \quad y'(0) = 0$

Solution:

Applying the *ZR-Transform* to both sides of the differential equation:

$ZR[y''] + 3ZR[y'] + 2ZR[y] = 0,$

Utilizing the derivative operational properties derived in Theorem 3.6:

$$\left(s^4\, ZR[y] - \frac{s}{\ln(s+1)}y(0) - \frac{1}{s\ln(s+1)}y'(0)\right) + 3\left(s^2\, ZR[y] - \frac{1}{s\ln(s+1)}y(0)\right) + 2ZR[y] = 0$$

Substituting the given initial conditions $y(0) = 1,$ and $y'(0) = 0$:

$$\left(s^4\, ZR[y] - \frac{s}{\ln(s+1)}\right) + 3\left(s^2\, ZR[y] - \frac{1}{s\ln(s+1)}y(0)\right) + 2ZR[y] = 0$$

Factoring out $ZR[y]$ gives:

$$ZR[y]\left(s^4+3s^2+2\right) = \frac{s}{\ln(s+1)} + \frac{3}{s\ln(s+1)} = \frac{s^2+3}{s\ln(s+1)}$$

Rearranging terms to solve algebraically for $ZR[y]$:

$$ZR[y] = \frac{s^2+3}{s\ln(s+1)(s^2+1)(s^2+1)}$$

Resolving the term $\frac{s^2+3}{(s^2+1)(s^2+1)}$ into partial fractions:

$$\frac{s^2+3}{(s^2+1)(s^2+1)} = \frac{2}{s^2+1} - \frac{1}{s^2+2}$$

Thus:

$$ZR[y] = \frac{2}{(s^2+1)s\ln(s+1)} - \frac{1}{(s^2+2)s\ln(s+1)}$$

Taking the inverse *ZR-Transform* yields the exact analytical solution:

$y(t) = 2e^{-t} - e^{-2t}$

4. Conclusions

In this work, we established the mathematical framework and operational calculus of the *ZR-Transform*. We derived its fundamental properties such as linearity, shifting, convolution, derivatives, and periodic functions and established direct duality relations with existing integral transforms. Applying the transform to several initial value problems showed that it simplifies linear differential equations into straightforward algebraic systems with exact closed-form solutions. Future work will

extend this framework to fractional differential equations, partial differential equations, and systems with singular kernels.

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