Time-Fractional Differential Equations Heat Conduction in Orthogonal Curvilinear and Cylindrical Systems

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ABSTRACT

In this paper, we derive a fractional Fourier law from fractional-order analysis. Based on the fractional Fourier law, we construct a fractional heat conduction equation with fractional derivatives in a general orthogonal curvilinear coordinate system. Fractional heat conduction equations in other orthogonal coordinate systems can be easily obtained as special cases. Furthermore, we obtain generalized H-function solutions to the fractional heat conduction equation using the method of integral transformation. heat conduction equation for $0 < \propto \leq 1$ interpolates the standard heat conduction equation $(\propto = 1)$ and the local heat conduction equation $(\propto \to 0)$. The resulting temperature fields are expressed compactly through Bessel functions together with generalized Mittag-Leffler and H-functions, offering analytic formulas that remain well-behaved for numerical evaluation and reveal transparent parameter dependence. the results provide a rigorous, geometry-consistent foundation and solution toolkit for modeling anomalous heat transport, with immediate implications for complex, inhomogeneous, or size-dependent thermal processes where classical Fourier conduction is inadequate. Finally, numerical results for different orders of fractional derivatives are presented graphically.

1. Introduction

Fractional-order analysis now appears across natural science, engineering, biology, and economics. Recent work on fractional integro-differential equations has pushed the field forward in each of these areas [1-12]. Its appeal in modeling transport through complex, heterogeneous, and disordered media comes largely from its ability to represent memory effects in a natural way [13-18]. Numerical studies have also found that heat transport in many one-dimensional systems with exact momentum conservation departs from Fourier's law: thermal conductivity changes with system size [19]. Li and Wang [20,21] derived a simple relation between anomalous thermal conductivity and anomalous diffusion. Povstenko [22] examined a fractional heat-conduction equation in which the ordinary diffusion equation's partial derivatives are replaced with derivatives of non-integer order. The fractional-order time diffusion equation with two spatial coordinates has been examined by many authors; see [23-26] and the references cited there. For angularly symmetric cases, Povstenko [23] studied the two-dimensional time-fractional diffusion-wave equation in a half-space. Özdemir [25] derived analytical and numerical solutions for the axisymmetric diffusion-wave equation involving two spatial coordinates. Özdemir and Karadeniz [26] examined the corresponding axisymmetric problem in a finite-length cylinder and calculated the fractional derivatives with the Grünwald-Letnikov approximation. Building on that work, we derive a fractional Fourier law from fractional analysis. Section 2 then uses this law to obtain the three-dimensional homogeneous fractional heat-conduction equation in a general orthogonal curvilinear coordinate system. Section 3 derives an analytical solution to the fractional heat-conduction equation in cylindrical coordinates using an integral transform. For $0<\propto\leq 1$, the equation interpolatess $(\propto= 1)$ and the local heat conduction

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equation ($\propto \rightarrow 0$). the fractional derivatives are presented graphically. In Section 4, we present our conclusions.

2- Fractional heat conduction equation in the general orthogonal curvilinear coordinate system

The classical theory of heat conduction is based on the Fourier law

$$q = -k \, grad \, T, \tag{1}$$

Relating the heat flux vector to the temperature gradient, where k is the thermal conductivity. Energy conservation is expressed locally by

$$-\nabla q(t) = \rho C_b \frac{\partial T(r,t)}{\partial t}, \tag{2}$$

where C_b is the specific heat capacity at constant pressure and ρ is density. In the theory of heat conduction proposed by Norwood [27], the corresponding generalization of the Fourier law

$$q(t) = -k \int_0^t \alpha((t-\tau) \, grad \, T(\tau) \, d\tau, \tag{3}$$

when kernel $\alpha(t-\tau) = 1$ leads to the wave equation for the temperature and to the thermoelasticity without energy dissipation [28]. The time-non-local dependence between the heat flux vector and the temperature gradient with the `longtail' power kernel [29]

$$q(t) = -\frac{k}{\Gamma(\propto)} \int_0^t (t-\tau)^{\propto -1} \frac{\partial}{\partial \tau} \, grad \, T(\tau) d\tau, \quad 0 < \propto \leq 1. \tag{4}$$

Considering the definitions of Caputo fractional derivative [30]

$${}_0^C D_t^{\propto} f(t) = \frac{1}{\Gamma(n-\propto)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\propto+1-n}} d\tau, \quad (n-1 < \propto \leq n), \tag{5}$$

Eq. (4) becomes

$$q(t) = -k\left({}_0^C D_t^{1-\propto} grad \, T\right), \quad 0 < \propto \leq 1. \tag{6}$$

The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equation. The Laplace transform form of Caputo fractional derivative is [31]

$$\int_0^{\infty} e^{-St} \, {}_0^C D_t^{\propto} f(t) dt = S^{\propto} \bar{f}(S) - \sum_{k=0}^{n-1} S^{\propto -k-1} f^{(k)}(0), \quad (n-1 < \propto \leq n). \tag{7}$$

The Laplace transform rule for the Caputo derivative allows the use of initial values of classical integer derivatives with known physical interpretations, while the Laplace transform rule for the Riemann-

Liouville fractional derivative requires the knowledge of initial values of fractional integrals and function derivatives (see Ref. [31], pp. 79-81). It should be emphasized that for zero initial conditions, the Riemann-Liouville and Caputo fractional derivatives yield the same result. We call Equation (6) the time fractional Fourier law. Equation (6) converges to the classical Fourier law, i.e., Equation. (1) for $\propto \rightarrow 1$.

Let the functional relationship between orthogonal curvilinear coordinates dollar (u_1, u_2, u_3) and the rectangular coordinates (x, y, z) be given as

$$x = X(u_1, u_2, u_3), \quad y = Y(u_1, u_2, u_3), \quad z = Z(u_1, u_2, u_3). \tag{8}$$

Then, the gradient of temperature in the orthogonal curvilinear coordinate system is given by (see Ref. [32], pp. 9_11)

$$\nabla \mathcal{T} = u_1 \frac{1}{\mathfrak{b}_1}\frac{\partial \mathcal{T}}{\partial u_1} + u_2 \frac{1}{\mathfrak{b}_2}\frac{\partial \mathcal{T}}{\partial u_2} + u_3 \frac{1}{\mathfrak{b}_3}\frac{\partial \mathcal{T}}{\partial u_3}. \tag{9}$$

Here, the coefficients $\mathfrak{b}_1; \mathfrak{b}_2 \ and \ \mathfrak{b}_3$ are called the scale factors and can be evaluated by

$$\mathfrak{b}_i^2 = \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2, \qquad i = 1,2,3 \tag{10}$$

and $u_1; u_2 \ and \ u_3$ are the unit direction vectors in the $u_1; u_2 \ and \ u_3$ directions in general orthogonal curvilinear coordinate system. The heat flux vector q becomes

$$q = -k\left({}_0^C\mathcal{D}_t^{1-\propto} \sum_{i=1}^{3} \overrightarrow{u_i} \frac{1}{\mathfrak{b}_i}\frac{\partial \mathcal{T}}{\partial u_i}\right), \qquad 0 < \propto \leq 1. \tag{11}$$

The divergence of the heat flux vector q is given by

$$\nabla q = -\frac{1}{\mathfrak{b}}\left[\frac{\partial}{\partial u_1}\left({}_0^C\mathcal{D}_t^{1-\propto}\frac{\mathfrak{b}}{\mathfrak{b}_1^2}\frac{\partial \mathcal{T}}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left({}_0^C\mathcal{D}_t^{1-\propto}\frac{\mathfrak{b}}{\mathfrak{b}_2^2}\frac{\partial \mathcal{T}}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left({}_0^C\mathcal{D}_t^{1-\propto}\frac{\mathfrak{b}}{\mathfrak{b}_3^2}\frac{\partial \mathcal{T}}{\partial u_3}\right)\right], \tag{12}$$

here $\mathfrak{b} = \mathfrak{b}_1\,\mathfrak{b}_2\,\mathfrak{b}_3$. the general orthogonal curvilinear coordinate system is now obtained by substituting the result given by Eq. (12) into Eq. (2)

$$\frac{1}{\mathfrak{b}}\left[\frac{\partial}{\partial u_1}\left(\frac{\mathfrak{b}}{\mathfrak{b}_1^2}\frac{\partial \mathcal{T}}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{\mathfrak{b}}{\mathfrak{b}_2^2}\frac{\partial \mathcal{T}}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{\mathfrak{b}}{\mathfrak{b}_3^2}\frac{\partial \mathcal{T}}{\partial u_3}\right)\right] = \frac{1}{\alpha}\,{}_0^C\mathcal{D}_t^{\propto}\mathcal{T}, \tag{13}$$

where the thermal conductivity k is assumed to be constant and $\alpha = \frac{k}{\rho \mathcal{C}_{\mathfrak{b}}}$ is thermal diffusivity. By taking the relationships between the cylindrical coordinates and the rectangular coordinates, and by utilizing Eq. (10), the scale factors are determined as

$$\mathfrak{b}_1 = 1, \ \ \mathfrak{b}_2 = r, \ \ \mathfrak{b}_3 = 3, \ \ \mathfrak{b}_4 = r. \tag{14}$$

Let

$$u_1 = r, \qquad u_2 = \emptyset, \qquad u_3 = z. \tag{15}$$

Then substituting Eqs. (14), (15) into Eq. (13), we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathcal{T}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \mathcal{T}}{\partial \emptyset^2} + \frac{\partial^2 \mathcal{T}}{\partial z^2} = \frac{1}{\alpha}\,{}_0^{\mathcal{C}}\mathcal{D}_t^{\propto}\mathcal{T}, \tag{16}$$

where $\mathcal{T} \equiv \mathcal{T}(r, \emptyset, z, t)$. This is the three-dimensional fractional heat conduction equation in the cylindrical coordinate system.

3- Solution of fractional heat conduction equation in the cylindrical coordinate system

In this section we consider the solution of the following time fractional heat conduction problem for an infinite medium in which the temperature varies with both r and $\emptyset$ variables.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathcal{T}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \mathcal{T}}{\partial \emptyset^2} = \frac{1}{\alpha}\left({}_0^{\mathcal{C}}\mathcal{D}_t^{\propto}\mathcal{T}\right), \quad 0 \le r < \infty, \quad 0 \le \emptyset \le 2\pi, \quad t > 0, \tag{17}$$

$$0 < \propto \le 1$$

$$\mathcal{T}(r, \emptyset, 0) = \mathcal{F}(r, \emptyset), \quad 0 \le r < \infty, \tag{18}$$

and subject to the condition that temperature remains finite at $r = 0$. Applying the Laplace integral transform of Caputo fractional derivative (Eq. (7)) with respect to time t and the integral transformer

$$\tilde{\mathcal{T}}(r, \upsilon, t) = \int_0^{2\pi} cos\upsilon(\emptyset - \emptyset')\qquad \mathcal{T}(r, \emptyset', t)d\emptyset'$$

with respect to $\emptyset$ variable yields

$$\mathcal{S}^{\propto}\overset{\simeq}{\mathcal{T}} - \mathcal{S}^{\propto-1}\tilde{\mathcal{F}}(r, \upsilon) = \alpha\left(\frac{\partial^2 \overset{\simeq}{\mathcal{T}}}{\partial r^2} + \frac{1}{r}\frac{\partial \overset{\simeq}{\mathcal{T}}}{\partial r} - \frac{\upsilon^2}{r^2}\overset{\simeq}{\mathcal{T}}\right), \tag{19}$$

$$\overset{\simeq}{\mathcal{T}}(r, \upsilon, 0) = \tilde{\mathcal{F}}(r, \upsilon), \qquad 0 \le r < \infty. \tag{20}$$

The Hankel transform with respect to the spatial coordinate r, we get

$$\overset{\approx}{\bar{\mathcal{T}}} = \frac{\mathcal{S}^{\propto-1}\overset{\widehat{\simeq}}{\mathcal{F}}(\mathfrak{B}, \upsilon)}{\mathcal{S}^{\propto} + \alpha\mathfrak{B}^2}. \tag{21}$$

The resulting transform of the temperature is successively inverted by the inversion transform. Then the solution of the problems (17), (18) becomes

$$\mathcal{T}(r,\emptyset,t)=\frac{1}{\pi}\sum_{v}\int_{\mathfrak{B}=0}^{\infty}\mathfrak{B}\mathcal{J}_{v}(\mathfrak{B}r)\ell^{-1}\left[\frac{\mathcal{S}^{\propto-1}\widehat{\widetilde{\mathcal{F}}}(\mathfrak{B},v)}{\mathcal{S}^{\propto}+\alpha\mathfrak{B}^{2}}\right]d\mathfrak{B}$$
$$=\frac{1}{\pi}\sum_{v}\int_{\mathfrak{B}=0}^{\infty}\mathfrak{B}\mathcal{J}_{v}(\mathfrak{B}r)\mathcal{E}_{\propto,1}\left(-\alpha\mathfrak{B}^{2}t^{\propto}\right)\widehat{\widetilde{\mathcal{F}}}(\mathfrak{B},v)d\mathfrak{B}, \tag{22}$$

where $v=0,1,2,3,\ldots$ and replace π by 2π for $v=0,\mathcal{J}_{v}$ is the Bessel function of the first kind of order$v$,

$$\widehat{\widetilde{\mathcal{F}}}=\int_{r'=0}^{\infty}\int_{\emptyset'=0}^{2\pi}r'\,\mathcal{J}_{v}(\mathfrak{B}r')cosv(\emptyset-\emptyset')\mathcal{F}(r',\emptyset')d\emptyset'dr', \tag{23}$$

and $\mathcal{E}_{\propto,\lambda}(z)$ is the generalized Mittag-Leffler function defined by Ref. [31]

$$\mathcal{E}_{\propto,\lambda}(z)=\sum_{\hbar=0}^{\infty}\frac{\mathcal{Z}^{\hbar}}{\Gamma(\propto\hbar+\lambda)},\qquad(\propto>0,\quad\lambda>0)$$

Here, we used an important formula of the Laplace transform of generalized Mittag-Leffler function

$$\mathcal{L}^{-1}\left\{\frac{\mathcal{S}^{\propto-\lambda}}{(\mathcal{S}^{\propto}\mp\mathcal{C})};\,t\right\}=t^{\lambda-1}\mathcal{E}_{\propto,\lambda}(\pm\mathcal{C}t^{\propto}).$$

Substituting the Eq. (23) into Eq. (22), we have the solution of fractional heat conduction equation in the cylindrical coordinate system

$$\mathcal{T}(r,\emptyset,t)=\frac{1}{\pi}\sum_{v}\int_{r'=0}^{\infty}\int_{\emptyset'=0}^{2\pi}r'\,COS\,v(\emptyset$$
$$-\emptyset')\mathcal{F}(r',\emptyset')\left[\int_{\mathfrak{B}=0}^{\infty}\mathfrak{B}\mathcal{J}_{v}(\mathfrak{B}r')\mathcal{E}_{\propto,1}(-\alpha\mathfrak{B}^{2}t^{\propto})d\mathfrak{B}\right]d\emptyset'dr'. \tag{24}$$

For the special case of $\mathcal{F}(r,\emptyset)=\delta(r-r_{0})\delta(\emptyset-\emptyset_{0})/r$, the solution of Eq. (24) takes the form

$$\mathcal{T}(r,\emptyset,t)=\frac{1}{\pi}\sum_{v}COS\,v(\emptyset-\emptyset_{0})\left[\int_{\mathfrak{B}=0}^{\infty}\mathfrak{B}\mathcal{J}_{v}(\mathfrak{B}r)\mathcal{J}_{v}(\mathfrak{B}r_{0})\mathcal{E}_{\propto,1}(-\alpha\mathfrak{B}^{2}t^{\propto})d\mathfrak{B}\right], \tag{25}$$

or in non-dimensional quantities form

$$\mathcal{T}^{*}=\sum_{v}cos2\pi v\varphi\left[\int_{\eta=0}^{\infty}\eta\mathcal{J}_{v}(\eta\mathcal{R})\mathcal{J}_{v}(\eta)\mathcal{E}_{\propto,1}(-\mathcal{P}^{2}\eta^{2})d\eta\right], \tag{26}$$

Where

$$\mathcal{T}^{*}=\pi r_{0}^{2}\mathcal{T},\ \mathcal{R}=\frac{r}{r_{0}},\qquad\eta=\mathfrak{B}r_{0},\qquad\mathcal{p}=\sqrt{\alpha t^{\propto}/r_{0}^{2}},\qquad\varphi=\frac{\emptyset-\emptyset_{0}}{2\pi}.$$

In the axisymmetric case ($\upsilon = 0$) this solution corresponding to time fractional diffusion equation has been considered in Ref. [33]. In order to obtain the exact solution of above equation, we employ the procedure presented in Ref. [34]

$$\mathcal{E}^{(n)}_{\propto,\mu}(y) = \mathcal{H}^{1,1}_{1,2}\left[-y\,\middle|\begin{matrix}(0,1)\\(0,1),(1-(\propto n+\mu),\propto)\end{matrix}\right], \tag{27}$$

and the relation between Bessel function and H-function [35]

$$\mathcal{J}_{\upsilon}(z) = \mathcal{H}^{1,0}_{0,2}\left[\frac{\mathcal{Z}^2}{4}\middle|\left(\frac{\upsilon}{2},1\right),\left(-\frac{\upsilon}{2},1\right)\right]. \tag{28}$$

Utilizing the relations (27) and (28), the Eq. (26) can be expressed in the following form

$$\begin{aligned}&\mathcal{T}^*(\mathcal{R},\varphi,\mathcal{p})\\&=\sum_{\upsilon} cos2\pi\upsilon\varphi\int_{\eta=0}^{\infty}\eta\mathcal{J}_{\upsilon}(\eta\mathcal{R})\mathcal{H}^{1,0}_{0,2}\left[\frac{\eta^2}{4}\middle|\left(\frac{\upsilon}{2},1\right),\left(-\frac{\upsilon}{2},1\right)\right]\mathcal{H}^{1,1}_{1,2}\left[\mathcal{p}^2\eta^2\,\middle|\begin{matrix}(0,1)\\(0,1),(0,\propto)\end{matrix}\right]d\eta.\end{aligned} \tag{29}$$

Using the Hankel transform of the product of two H-function [35],

$$\begin{aligned}&\int_0^{\infty} x^{\rho-1}\,\mathcal{J}_{\upsilon}(\alpha x)\mathcal{H}^{\mathcal{M},\mathcal{N}}_{\mathcal{A},\mathcal{B}}\left[\mathcal{b}x^2\middle|\begin{matrix}(\alpha_{\mathcal{A}},\propto_{\mathcal{A}})\\(\mathcal{b}_{\mathcal{B}},\mathfrak{B}_{\mathcal{B}})\end{matrix}\right]\mathcal{H}^{\mathcal{M}_1,\mathcal{N}_1}_{\mathcal{C},\mathcal{D}}\left[\mathcal{C}x^2\middle|\begin{matrix}c_{\mathcal{C}},\mathcal{y}_{\mathcal{C}}\\d_{\mathcal{D}},\mathcal{S}_{\mathcal{D}}\end{matrix}\right]dx\\&=2^{\rho-1}\alpha^{-\rho}\mathcal{H}^{1,\mathcal{N},\mathcal{N}_1,\mathcal{M},\mathcal{M}_1}_{2,[\mathcal{A}:\mathcal{C}],0,[\mathcal{B}:\mathcal{D}]}\left[\begin{matrix}\frac{4\mathcal{b}}{\alpha^2}\\\frac{4\mathcal{C}}{\alpha^2}\end{matrix}\middle|\begin{matrix}\left(\frac{\rho+\upsilon}{2},1\right);&\left(\frac{\rho-\upsilon}{2},1\right)\\(\alpha_{\mathcal{A}},\propto_{\mathcal{A}});&(c_{\mathcal{C}},\mathcal{y}_{\mathcal{C}})\\-\ ;&-\\(\mathcal{b}_{\mathcal{B}},\mathfrak{B}_{\mathcal{B}})&(d_{\mathcal{D}},\mathcal{S}_{\mathcal{D}}).\end{matrix}\right],\end{aligned} \tag{30}$$

we obtain the solution of time fractional heat conduction equation

$$\mathcal{T}^* = \sum_{\upsilon}\frac{2cos2\pi\upsilon\varphi}{\mathcal{R}^2}\mathcal{H}^{1,0,1,1,1}_{2,[0:1],0,[2:2]}\left[\begin{matrix}\frac{1}{\mathcal{R}^2}\\\frac{4\mathcal{p}^2}{\mathcal{R}^2}\end{matrix}\middle|\begin{matrix}\left(\frac{2+\upsilon}{2},1\right)&;\left(\frac{2-\upsilon}{2},1\right)\\&;\ (0,1)\\-&;\ -\\\left(\frac{\upsilon}{2},1\right),\left(\frac{-\upsilon}{2},1\right)&;\ (0,1),(0.\propto)\end{matrix}\right], \tag{31}$$

where $\mathcal{H}^{m,n}_{\mathcal{p},\mathcal{q}}\left[z\middle|\begin{matrix}\alpha_{\mathcal{p}},\propto_{\mathcal{p}}\\\mathcal{b}_{\mathcal{q}},\mathfrak{B}_{\mathcal{q}}\end{matrix}\right]$ denotes H-function [35] and

$$\mathcal{H}\left[\begin{matrix}x\\y\end{matrix}\right]\mathcal{H}^{1,\mathcal{N},\mathcal{N}_1,\mathcal{M},\mathcal{M}_1}_{\mathcal{E},[\mathcal{A}:\mathcal{C}],\mathcal{F},[\mathcal{B}:\mathcal{D}]}\left[\begin{matrix}x\\y\end{matrix}\middle|(x)\right], \tag{32}$$

is the generalized H-function [35] . Let us analyze some particular cases [32,36]. In this case $\mathcal{E}_{1.1}(z) = \mathcal{E}_1(z) = \ell^z$. (26) becomes

$$\mathcal{T}^*(\mathcal{R},\varphi,\mathcal{p}) = \sum_{\upsilon} cos2\pi\upsilon\varphi\left[\int_{\eta=0}^{\infty}\eta\mathcal{J}_{\upsilon}(\eta\mathcal{R})\mathcal{J}_{\upsilon}(\eta)exp(-\mathcal{P}^2\eta^2)d\eta\right],$$

$$= \frac{1}{2p^2}\sum_{v} cos2\pi v\varphi \; exp\left(-\frac{\mathcal{R}^2+1}{4p^2}\right) J_v\left(\frac{\mathcal{R}}{2p^2}\right) \tag{33}$$

$$= \frac{r_0^2}{2\alpha t}\sum_{v} cosv(\emptyset-\emptyset_0) exp\left(-\frac{r^2+r_0^2}{4\alpha t}\right) J_v\left(\frac{r r_0}{2\alpha t}\right)$$

here we use the integral [37] $\int_0^\infty exp(-p^2x^2)\, J_v(\alpha x) J_v(bx) x dx = \frac{1}{2p^2} exp\left(-\frac{\alpha^2+b^2}{4p^2}\right) J_v\left(\frac{\alpha b}{2p^2}\right)$. The limiting case $\propto \to 0$ corresponds to Localized heat conduction. In this case

$$\lim_{\propto\to 0} \mathcal{E}_{\propto,1}(-p^2\eta^2) = \frac{1}{1+\alpha\eta^2/r_0^2}.$$

Eq. (26) becomes

$$\mathcal{T}^*(\mathcal{R},\varphi,p) = \sum_{v} cos2\pi v\varphi \left[\int_{\eta=0}^{\infty} \frac{\eta}{1+\alpha\eta^2/r_0^2} J_v(\eta\mathcal{R}) J_v(\eta) d\eta\right]$$
$$= \frac{1}{p^2}\sum_{v} cos2\pi v\varphi J_v\left(\frac{1}{p}\right) h_v\left(\frac{\mathcal{R}}{p}\right) = \frac{r_0^2}{\alpha}\sum_{v} cosv(\emptyset-\emptyset_0) J_v\left(\frac{r_0}{\sqrt{\alpha}}\right) h_v\left(\frac{r}{\sqrt{\alpha}}\right), \tag{34}$$

here we use the relation [37]

$$\int_0^\infty \frac{x J_v(\alpha x) J_v(bx)}{x^2+k^2} dx = J_v(bk) K_v(\alpha k), \qquad (\alpha > b),$$

where J_v and h_v are the modified first-order and second-order Bessel functions v, respectively. For a fixed time value $(t = 0{,}1)$, the larger the $\propto$, the higher the temperature $\mathcal{T}^*$ near the origin. Obviously, the temperature near the origin. However, the effect on the critical point temperature seems to be the opposite. In fact, for large values of $\mathcal{R}$, the asymptotic behavior of the solution (31) can be discussed. Using the asymptotic behavior of the generalized H-function (36) , we obtain the asymptotic expression of the solution (31)

$$\mathcal{T}^* \sim \mathcal{A}\sum_{v} \frac{cos2\pi v\varphi}{\mathcal{R}^2} o(|\mathcal{R}|^v), \tag{35}$$

where $\mathcal{A}$is constant.

4- Conclusions

In summary, this paper derives the fractional-order heat conduction equation in a general orthogonal curvilinear coordinate system. As special cases, the fractional-order heat conduction equation in rectangular, cylindrical, and spherical coordinate systems can be easily derived from the

general equation (13) by introducing appropriate scaling factors. Furthermore, solutions to the fractional-order heat conduction equation in cylindrical coordinates are obtained using the generalized H-function. The fractional-order heat conduction equation for the case $0 < \propto < 1$ is an interpolation of the standard heat conduction equation ($\propto = 1$) and the local heat conduction equation ($\propto \to 0$). Numerical results are also presented graphically.

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