Lie Point Symmetries and Exact Solution of The Time-Fractional MHD Boundary Layer System

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ABSTRACT

This study investigates the unsteady magneto hydrodynamic boundary layer flow and heat transfer of a viscoelastic fluid over a stretching sheet with a heat source, incorporating fractional-order Caputo derivatives to capture memory effects. We systematically derive the Lie point symmetries of the governing system using an extended jet-space and the Gazizov–Kasatkin–Lukashchuk (GKL) fractional prolongation framework. The determining equations reveal specific constraints on the temporal and spatial infinitesimal generators, yielding an infinite-dimensional abelian ideal and thermal subalgebras. Furthermore, we construct exact solutions corresponding to each symmetry operator—including spatial translations, scalings, spatial modulations, temperature rescalings, and fractional memory-dependent temporal evolutions. Comprehensive two-dimensional comparative profiles and three-dimensional surface and contour visualizations are presented to illustrate the influence of the magnetic parameter, fractional order, and memory kernels on the velocity and thermal fields.

1. Introduction and Backgropund

Viscoelastic fluids, such as polymer solutions, melts and many biological and industrial liquids, keep a memory of their past deformation: the stress does not depend only on the present rate of strain. Fractional calculus provides a natural and economical way to describe this behaviour, because a fractional derivative of order $\alpha \in (0,1)$ in the constitutive law interpolates between the elastic ($\alpha=0$) and the viscous ($\alpha=1$) response and carries the whole history of the motion in a single nonlocal operator [21, 14, 28]. Fractional Maxwell and Oldroyd-B models have been used for unsteady flows of viscoelastic fluids with magnetic field, slip and heat transfer [27, 28].

The system studied in this paper, (2.2) below, describes the unsteady boundary layer flow and heat transfer of an MHD viscoelastic fluid over an unsteady stretching sheet in the presence of thermal radiation and a heat source. It was derived and solved numerically by Shen, Zheng and Chen [23], who replaced the classical relaxation term of the viscoelastic constitutive law by a time-fractional derivative so that the flow and heat transfer pass from short-term to long-term memory; they used a finite difference scheme based on the L1 approximation and reported a delay-time effect on the velocity and temperature fields. Several closely related fractional viscoelastic boundary layer and heat transfer problems have also been published. Liu and Guo [17] studied the unsteady MHD flow of a generalized Maxwell fluid with radiation, a heat source and a time-fractional Fourier law, and solved it by the Laplace transform with numerical inversion. Chen, Zheng, Li and Sui [5] considered time–space fractional viscoelastic MHD flow and heat transfer over an accelerating plate with slip, using explicit finite differences and deriving exact solutions in limiting cases. Hanif [10] analysed the boundary layer flow and heat transfer of a fractional Maxwell fluid with a Cattaneo heat flux by a Crank–Nicolson scheme combined with the L1 approximation. All these works obtain approximate (numerical or transform-based) solutions; to the best of our knowledge, a group-theoretic analysis of the fractional boundary layer system (2.2) has not been carried out. Such an analysis produces, without any discretization, the full set of point symmetries, reduced equations and closed-form invariant solutions, which also serve as benchmarks for numerical schemes.

Sophus Lie's method looks for continuous groups of point transformations that map solutions of a differential equation into solutions. The infinitesimal generators are found from linear determining

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equations, and invariants of the generators give group-invariant (similarity) solutions through a reduction of the number of independent variables [20, 2, 19, 12]. The classical approach has been used widely for boundary layer flows over stretching sheets, see e.g. [9]. For fractional equations the main difficulty is that the derivative is nonlocal, so the usual prolongation to jet space is not enough. The first results are due to Buckwar and Luchko [3], who studied the invariance of a fractional diffusion equation under scaling transformations, and to Djordjevic and Atanackovic [6], who obtained similarity solutions of fractional heat conduction, Burgers and Korteweg–de Vries equations. Gazizov, Kasatkin and Lukashchuk [7, 8] introduced the fractional prolongation of the generator, which extends Lie's infinitesimal criterion to equations with Riemann–Liouville or Caputo derivatives; a necessary condition is that the lower limit of the derivative is an invariant of the group (for derivatives with lower limit t=0, τ|_(t=0)=0). A foundational justification of the construction was given by Leo, Sicuro and Tempesta [15, 16]. Zhang and Zheng [26] derived an explicit prolongation formula for systems of multi-dimensional time-fractional equations and showed that the symmetry generator has a simple unified structure, and Jefferson and Carminati [13] developed symbolic software for the computation. The method has been applied to many fractional models, for example Burgers and KdV equations [22], fifth-order KdV [24], Sharma–Tasso–Olver [25] and two classes of fractional partial differential equations [4], and to fractional ordinary differential equations [1]. Invariant solutions of fractional PDEs reduce them to fractional ODEs, typically with Erdélyi–Kober operators [22, 25]; conservation laws are treated in [18]. A recent monograph on the subject is [11].

This paper is organized as follows. Section 2 introduces the mathematical model for the unsteady magnetohydrodynamic (MHD) boundary layer flow and heat transfer of a viscoelastic fluid over a stretching sheet, incorporating time-fractional Caputo derivatives to describe short-term and long-term memory effects. It also establishes the foundational equivalence transformations and prolongation framework. Section 3 focuses on the systematic determination of the Lie point symmetries for the time-fractional governing system, employing the Gazizov–Kasatkin–Lukashchuk (GKL) framework and explicit standing assumptions to solve the overdetermined system of determining equations, ultimately classifying the resulting Lie algebra structure. Finally, Section 4 explores the symmetry-generated exact solutions across five distinct cases—spanning spatial translations, scaling transformations, spatial modulations, temperature scaling, and additive thermal transformations governed by fractional memory equations—accompanied by a thorough graphical analysis using both two-dimensional comparative curves and three-dimensional surface and contour representations.

2. Mathematical model and symmetry analysis

We consider the unsteady MHD flow of a viscoelastic fluid and the accompanying heat transfer over an unsteady stretching sheet in the presence of a heat source; the pressure gradient is ignored. To account for the passage of the flow and heat transfer process from short-term to long-term memory, the relationship between stress and strain of the viscoelastic medium is described by fractional derivatives. The two-dimensional unsteady fractional boundary layer equations areBulleted lists may be included and should look like this:

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0, \\ (1+\lambda\, D_t^{\alpha})\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=\nu\dfrac{\partial^2 u}{\partial y^2}-\dfrac{\sigma B_0^2}{\rho}\,u, \\ (1+\lambda\, D_t^{\alpha})\dfrac{\partial T}{\partial t}+u\dfrac{\partial T}{\partial x}+v\dfrac{\partial T}{\partial y}=\dfrac{k}{\rho c_p}\dfrac{\partial^2 T}{\partial y^2}-\dfrac{1}{\rho c_p}\dfrac{\partial q_r}{\partial y}+\dfrac{Q_0}{\rho c_p}\,(T-T_\infty), \end{cases} \tag{2.1}$$

where u, v are the velocity components in the x- (along the sheet) and y-directions, T the temperature, ν the kinematic viscosity, ρ the density, $c_\rho$the specific heat, k the thermal conductivity, λ the relaxation-time (viscoelastic) parameter and $\alpha \in (0,1)$ the fractional order. The term νu_{yy} is the viscous diffusion term, $(\sigma B_0^2/\rho)\,u$ is the magnetic-field (Lorentz force) term, $(q_r)_y$ is the radiation term and $Q_0(T-T_\infty)$ describes the heat source in the boundary layer, with σ the electrical

conductivity, B_0 the applied magnetic field and $T_\infty$the ambient temperature. The operator $D_t^\alpha$is the Caputo derivative with lower limit $t = 0$,

$$D_t^\alpha w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\, w'(s)\, ds, \qquad 0 < \alpha < 1.$$

The system is completed by initial conditions and by boundary conditions on the stretching sheet and in the far field, as in [23]; they are not needed for the computation of the Lie point symmetries, which is local, and are used later when the invariant solutions are selected.

Using compact subscript notation for partial derivatives (where $u_x = \frac{\partial u}{\partial x}$, $u_t = \frac{\partial u}{\partial t}$, and $u_{yy} = \frac{\partial^2 u}{\partial y^2}$), the governing system of partial differential equations with time-fractional derivatives of order α is expressed as follows:

$$\begin{cases} \Delta_1 & := u_x + v_y = 0, \\ \Delta_2 & := (1 + \lambda D_t^\alpha) u_t + u\, u_x + v\, u_y = \nu\, u_{yy} - \dfrac{\sigma B_0^2}{\rho}\, u, \\ \Delta_3 & := (1 + \lambda D_t^\alpha) T_t + u\, T_x + v\, T_y = \dfrac{k}{\rho c_p}\, T_{yy} - \dfrac{1}{\rho c_p}\, (q_r)_y + \dfrac{Q_0}{\rho c_p}\, (T - T_\infty) \end{cases} \tag{2.2}$$

where u and v represent the velocity components along the x and y directions, respectively, T is the fluid temperature, λ is the relaxation time, D_t^α denotes the time-fractional derivative operator, ν is the kinematic viscosity, σ is the electrical conductivity, B_0 is the magnetic field strength, ρ is the fluid density, k is the thermal conductivity, c_p is the specific heat at constant pressure, q_r is the radiative heat flux, Q_0 is the volumetric heat generation/absorption coefficient, and T_∞ is the ambient temperature.

An equivalence transformation for the system (2.2) is defined by invertible transformations of the independent and dependent variables of the form:

$$\bar{t} = \Phi(t, u, v, T), \quad \bar{x} = \Theta_x(t, x, y, u, v, T), \quad \bar{y} = \Theta_y(t, x, y, u, v, T), \tag{2.3}$$

and for the dependent fields:

$$\bar{u} = \Psi_u(t, u, v, T), \quad \bar{v} = \Psi_v(t, u, v, T), \quad \bar{T} = \Psi_T(t, u, v, T), \tag{2.4}$$

such that the system retains the same structural form in the new variables:

$$\begin{cases} \bar{\Delta}_1 & := \bar{u}_{\bar{x}} + \bar{v}_{\bar{y}} = 0, \\ \bar{\Delta}_2 & := (1 + \lambda D_{\bar{t}}^\alpha) \bar{u}_{\bar{t}} + \bar{u}\, \bar{u}_{\bar{x}} + \bar{v}\, \bar{u}_{\bar{y}} = \nu\, \bar{u}_{\bar{y}\bar{y}} - \dfrac{\sigma B_0^2}{\rho}\, \bar{u}, \\ \bar{\Delta}_3 & := (1 + \lambda D_{\bar{t}}^\alpha) \bar{T}_{\bar{t}} + \bar{u}\, \bar{T}_{\bar{x}} + \bar{v}\, \bar{T}_{\bar{y}} = \dfrac{k}{\rho c_p}\, \bar{T}_{\bar{y}\bar{y}} - \dfrac{1}{\rho c_p}\, (\bar{q}_r)_{\bar{y}} + \dfrac{Q_0}{\rho c_p}\, (\bar{T} - \bar{T}_\infty) \end{cases} \tag{2.5}$$

where the barred quantities are new functions of the new arguments. If the form remains unaltered, the transformation is called an admitted equivalence transformation of the system (2.2).

A one-parameter group of transformations can be described by its infinitesimal operator. For equivalence transformations, it has the following complete general form:

$$\boldsymbol{X} = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \eta^u \frac{\partial}{\partial u} + \eta^v \frac{\partial}{\partial v} + \eta^T \frac{\partial}{\partial T}, \tag{2.6}$$

where the components $\xi^t, \xi^x, \xi^y, \eta_u, \eta_v,$ and η_T are functions of (t, x, y, u, v, T). According to the prolongation theory for fractional derivatives, the action of infinitesimal transformations on the variables is defined by prolongation formulas, where the extended infinitesimal terms can be represented via fractional derivatives and integrals using generalized Leibniz rules.

The determining equations for finding the coefficients of the infinitesimal operator are obtained by evaluating the prolongation conditions on the manifold defined by system (2.2):

$$\zeta^{\alpha}|_{\Delta_i=0} = 0, \quad i = 1,2,3. \tag{2.7}$$

By analogy with the algorithm for constructing coordinates of admitted operators, symmetries and equivalence transformations are sought from the appropriate ansatz class for the infinitesimal generators.

3. Lie point symmetries of the time-fractional MHD boundary layer system

The systematic determination of the Lie point symmetries for the time-fractional magnetohydrodynamic boundary layer system governed by $\Delta_1 = 0$, $\Delta_2 = 0$, and $\Delta_3 = 0$ is established through an extended jet-space analysis. Because the model features Caputo time-fractional derivatives $\mathrm{D}_t^{\beta} u$ and $\mathrm{D}_t^{\beta} T$ of order $\beta = 1 + \alpha$ with $\alpha \in (0,1)$, the classical prolongation apparatus is augmented via the Gazizov–Kasatkin–Lukashchuk (GKL) framework. The analysis relies on the explicitly defined standing assumptions (S1)–(S4), which provide the necessary closure for the nonlocal prolongation structures:

(S1) $\alpha \in (0,1)$, $\lambda \neq 0$ (ensuring the memory term is genuinely present with $\alpha\lambda \neq 0$), and $\nu \neq \kappa$.

(S2) The lower limit $t = 0$ temporal restrictions $\tau(0) = 0$ and $\tau_{tt} = 0$, which yield the linear form $\tau = c_1 t$.

(S3) The spatial infinitesimal components ξ^1 and ξ^2 depend exclusively on (x, y) such that $\xi_t = 0$ and $\xi_u = 0$.

(S4) The characteristic functions take the structured forms $\varphi^u = a_1 u + a_2 T + d_1$ and $\varphi^T = e_1 T + e_2 u + d_2$, where a_1, a_2, e_1, e_2 depend on (x, y) only and d_1, d_2 depend on (x, y, t), excluding transverse velocity terms in v to prevent $\mathrm{D}_t^{\beta} v$ from becoming an independent quantity, while φ^v is treated as a general function of (x, y, t, u, v, T).

Lemma 3.1. *Let* $\boldsymbol{X}$ *be the infinitesimal generator given in (2.6), with components* ξ^1, ξ^2, ξ^3 *along* x, y, t *and* η^w *along the dependent variables* $w \in \{u, v, T\}$ *of the system (2.2). Write* $(x^1, x^2, x^3) = (x, y, t)$. *For a multi-index* $J = (j_1, \ldots, j_k)$ *with* $j_l \in \{x, y, t\}$ *denote* $w_J = \partial^k w / \partial x^{j_1} \cdots \partial x^{j_k}$ *(with* $w_{\emptyset} = w$*). The total derivatives are*

$$\mathrm{D}_i = \partial_{x^i} + \sum_{w} \sum_{J} w_{Ji}\, \partial_{w_J}, \qquad i \in \{x, y, t\}, \tag{3.1}$$

that is,

$$
\begin{aligned}
D_x &= \partial_x + \sum_w \left(w_x\, \partial_w + w_{xx}\, \partial_{w_x} + w_{xy}\, \partial_{w_y} + w_{xt}\, \partial_{w_t} + \cdots \right), \\
D_y &= \partial_y + \sum_w \left(w_y\, \partial_w + w_{xy}\, \partial_{w_x} + w_{yy}\, \partial_{w_y} + w_{yt}\, \partial_{w_t} + \cdots \right), \\
D_t &= \partial_t + \sum_w \left(w_t\, \partial_w + w_{xt}\, \partial_{w_x} + w_{yt}\, \partial_{w_y} + w_{tt}\, \partial_{w_t} + \cdots \right),
\end{aligned}
$$

and $D_J := D_{j_1} \cdots D_{j_k}$. Define the characteristics

$$Q^w := \eta^w - \xi^1 w_x - \xi^2 w_y - \xi^3 w_t. \tag{3.2}$$

Then the coefficients of the classical prolongation $\mathrm{pr}\,\mathbf{X}$ are given by

$$\eta_J^w = D_J Q^w + \xi^1 w_{Jx} + \xi^2 w_{Jy} + \xi^3\, w_{Jt}, \tag{3.3}$$

and satisfy the recursion $\eta_{Ji}^{w} = \mathrm{D_i}\eta_J^{w} - \sum_{k=1}^{3}\left(\mathrm{D_i}\xi^{k}\right) w_{Jk}$. Moreover, let $\beta := 1 + \alpha$ be the total order of the nonlocal term, and let D_t^{β} denote the Riemann–Liouville derivative (the Caputo case is addressed in the proof). Suppose that (S2) $\xi^3 = c_1 t$; (S3) $\xi^1 = \xi^1(x, y)$ and $\xi^2 = \xi^2(x, y)$; and (S4) $\eta^{u} = a_1 u + a_2 T + d_1(x, y, t)$ and $\eta^{T} = e_1 T + e_2 u + d_2(x, y, t)$ with constants a_1, a_2, e_1, e_2. Then the GKL fractional prolongation components *are*

$$
\begin{aligned}
\eta^{u,\beta} &= a_1 f_u + a_2 f_T + B_1 - (1+\alpha)\, c_1 f_u, \\
\eta^{T,\beta} &= e_1 f_T + e_2 f_u + B_2 - (1+\alpha)\, c_1 f_T,
\end{aligned} \tag{3.4}
$$

where $f_u := D_t^{\beta} u$, $f_T := D_t^{\beta} T$ and $B_i := D_t^{\beta} d_i$, $i = 1,2$.

Proof. We first establish (3.3) by induction on $|J|$. For $J = \emptyset$, $\eta^w = Q^w + \sum_{k=1}^{3} \xi^k\, w_k$ by the definition (3.2), so the formula reproduces the generator component η^w itself. Assume that it holds for J, that is, $\eta_J^w = \mathrm{D}_J Q^w + \sum_k \xi^k\, w_{Jk}$. Requiring that $\mathrm{pr}\,\mathbf{V}$ preserve the contact conditions gives the recursion $\eta_{Ji}^w = \mathrm{D}_i \eta_J^w - \sum_k \left(\mathrm{D}_i \xi^k\right) w_{Jk}$. Since $\mathrm{D}_i w_{Jk} = w_{Jki}$,

$$
\begin{aligned}
\eta_{Ji}^w &= \mathrm{D}_i \mathrm{D}_J Q^w + \sum_k \left(\mathrm{D}_i \xi^k\right) w_{Jk} + \sum_k \xi^k\, w_{Jki} - \sum_k \left(\mathrm{D}_i \xi^k\right) w_{Jk} \\
&= \mathrm{D}_{Ji} Q^w + \sum_k \xi^k\, w_{Jik},
\end{aligned}
$$

because the total derivatives commute and mixed partial derivatives are symmetric. This is (3.3) for the multi-index Ji, which completes the induction. For the fractional extension, let $n - 1 < \beta < n$ and

$$\mathrm{D}_t^{\beta} w = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1}\, w(x, y, s)\, \mathrm{d}s.$$

For a generator satisfying $\xi^3(0) = 0$, the Gazizov–Kasatkin–Lukashchuk extension of $\mathrm{pr}\,\mathbf{X}$ to $\mathrm{D}_t^{\beta} w$ has coefficient

$$\eta^{w,\beta} = D_t^{\beta}\eta^{w} + \sum_{i=1}^{2}\left[\xi^{i}\, D_t^{\beta}\left(w_{x^i}\right) - D_t^{\beta}\left(\xi^{i} w_{x^i}\right)\right] + D_t^{\beta}\left(D_t(\xi^{3})\, w\right) - D_t^{\beta+1}(\xi^{3} w) + \xi^{3}\, D_t^{\beta+1} w, \tag{3.5}$$

where $\mathrm{D}_t^{\beta}\eta^{w}$ is evaluated by the fractional chain rule. The condition $\xi^{3}(0) = 0$ expresses the invariance of the initial manifold $t = 0$ and holds under (S2); for the Caputo derivative the same formula applies, since the additional boundary terms vanish when $\xi^{3}(0) = 0$.

By (S3), ξ^{1} and ξ^{2} do not depend on t (nor, through u, T, on the dependent variables), so they act as constants for D_t^{β} and $\mathrm{D}_t^{\beta}\left(\xi^{i} w_{x^i}\right) = \xi^{i}\mathrm{D}_t^{\beta}\left(w_{x^i}\right)$; the bracket in (3.5) therefore vanishes. By (S2), $\mathrm{D}_t\xi^{3} = c_1$, hence $\mathrm{D}_t^{\beta}(\mathrm{D}_t(\xi^{3})w) = c_1\mathrm{D}_t^{\beta} w$. The generalized Leibniz rule

$$\mathrm{D}_t^{\gamma}(fg) = \sum_{n=0}^{\infty}\binom{\gamma}{n}\mathrm{D}_t^{\gamma-n} f\ \mathrm{D}_t^{n} g$$

with $g = \xi^{3} = c_1 t$ (so $\mathrm{D}_t^{n}\xi^{3} = 0$ for $n \geq 2$) and $\gamma = \beta + 1$ yields

$$\mathrm{D}_t^{\beta+1}(\xi^{3} w) = c_1 t\, \mathrm{D}_t^{\beta+1} w + (\beta + 1)\, c_1\, \mathrm{D}_t^{\beta} w.$$

Substituting into (3.5), the terms $\pm c_1 t\, \mathrm{D}_t^{\beta+1} w$ cancel and

$$\eta^{w,\beta} = \mathrm{D}_t^{\beta}\eta^{w} + c_1\mathrm{D}_t^{\beta} w - (\beta + 1)c_1\mathrm{D}_t^{\beta} w = \mathrm{D}_t^{\beta}\eta^{w} - \beta\, c_1\, \mathrm{D}_t^{\beta} w. \tag{3.6}$$

With $\beta = 1 + \alpha$ the coefficient of $\mathrm{D}_t^{\beta} w$ is $-(1 + \alpha)c_1$. As a consistency check, $\beta = 1$ reproduces the classical $\eta_t^{w} = \mathrm{D}_t\eta^{w} - c_1 w_t$.

Finally, by (S4) the components η^{u} and η^{T} are affine in (u, T) with constant coefficients, so the nonlinear terms of the fractional chain rule vanish and D_t^{β} acts linearly:

$$\mathrm{D}_t^{\beta}\eta^{u} = a_1\mathrm{D}_t^{\beta} u + a_2\mathrm{D}_t^{\beta} T + \mathrm{D}_t^{\beta} d_1 = a_1 f_u + a_2 f_T + B_1, \qquad \mathrm{D}_t^{\beta}\eta^{T} = e_1 f_T + e_2 f_u + B_2.$$

Inserting these into (3.6) with $w = u$ and $w = T$ gives (3.4). Every term is a linear combination of f_u, f_T and the known functions B_1, B_2, so the prolongation closes on the jet variables of the system and no new independent quantities appear. □

Lemma 3.2. Let $\boldsymbol{X}$ be the infinitesimal generator given in (2.6), with $\xi^{3} = c_1 t$ as in (S2), and let the standing assumption (S1), namely $\alpha\lambda \neq 0$ and $\nu \neq \kappa$, hold. Then the overdetermined system of determining equations obtained from the invariance condition

$$pr\, \boldsymbol{X}(\Delta)|_{\Delta=0} = 0$$

for the system (2.2), with the prolongation coefficients of Lemma 3.1, reduces uniquely to the following. The temporal scaling coefficient vanishes, $\mathrm{c}_1 = 0$, so that $\xi^{3} \equiv 0$; that is, the memory term excludes both time translations and time scalings. The spatial components are

$$\xi^{1} = Ax + x_0, \qquad \xi^{2} = Z(x), \tag{3.7}$$

with constants A, x_0 and an arbitrary smooth function Z, and the transverse velocity characteristic is

$$\eta^v = Z'(x)\,u. \tag{3.8}$$

The temperature characteristic has the form

$$\eta^T = E\,T + F(t), \tag{3.9}$$

with a constant E and a memory function F(t) governed by the linear fractional differential equation

$$F'(t) + \lambda\, D_t^{\beta} F(t) - Q_h\, F(t) = Q_h\, T_\infty\, E, \qquad \beta = 1 + \alpha. \tag{3.10}$$

Here D_t^{β} is the Caputo derivative, and the general solution of (3.10) is

$$F(t) = -T_\infty E + F_0(t), \tag{3.11}$$

where $F_0(t)$ solves the homogeneous fractional memory equation

$$F_0{}'(t) + \lambda\, D_t^{\beta} F_0(t) - Q_h\, F_0(t) = 0. \tag{3.12}$$

Proof. Substituting the prolongation coefficients of Lemma 3.1 into the invariance condition and eliminating the highest-order derivatives by means of the governing system (2.2), one obtains an expression that is polynomial in the remaining independent jet variables u, v, T, their spatial derivatives, and the nonlocal variables $f_u = D_t^{\beta} u$ and $f_T = D_t^{\beta} T$. Since these variables can be prescribed independently, the coefficient of each monomial must vanish, which produces the overdetermined system. Because of the term $-(1+\alpha)c_1 f_u$ in (3.4), the splitting with respect to the nonlocal jet variables yields the determining equation $\alpha\lambda c_1 = 0$. Since $\alpha\lambda \neq 0$ by (S1), it follows that $c_1 = 0$, hence $\xi^3 = c_1 t \equiv 0$.

With $c_1 = 0$, the splitting with respect to u, v and T removes all higher-order couplings between the components. The condition $\nu \neq \kappa$ in (S1) allows the coefficients of the diffusion terms of the momentum and energy equations to be separated, so that the constraints on $\xi^1, \xi^2, \eta^u, \eta^v, \eta^T$ decouple. The remaining constraints on the spatial components force ξ^1 to be affine in x and ξ^2 to depend on x alone, which gives (3.7), while the continuity constraint yields $\eta^v = Z'(x)u$, which is (3.8). The constants A and E appear as the free scaling parameters of the spatial and temperature components, respectively.

It remains to analyse the energy constraint $\mathcal{C}_3 = 0$. With $\eta^T = ET + F(t)$ and $c_1 = 0$, the prolongation coefficients reduce to $\eta_t^T = ET_t + F'(t)$ and $\eta^{T,\beta} = E\, D_t^{\beta} T + D_t^{\beta} F$, by (3.4) with $e_1 = E$, $e_2 = 0$ and $d_2 = F$. Using the energy equation to replace $T_t + \lambda\, D_t^{\beta} T$ and collecting the terms that depend on t alone, the constraint becomes exactly (3.10). Since the Caputo derivative annihilates constants, $D_t^{\beta}(-T_\infty E) = 0$, and substituting $F = -T_\infty E + F_0$ into (3.10) gives (3.12). Conversely, every solution F_0 of (3.12) produces a solution of (3.10), so (3.11) is the general solution. Equation (3.12) is linear with constant coefficients, so F_0 is determined by its initial data, for example through the Laplace transform. This establishes the stated form of the characteristics and the uniqueness of the reduction. □

Theorem 3.3. Let $\alpha \in (0,1)$, $\lambda \neq 0$, and $\nu \neq \kappa$. Suppose that the determining equations of the time-fractional MHD boundary-layer system admit the infinitesimal solution

$$\begin{aligned} \xi^1 &= 0, \\ \xi^2 &= Ax + x_0, \qquad \xi^3 = Z(x), \\ \eta^u &= Au, \qquad \eta^v = Z'(x)u, \\ \eta^T &= E(T - T_\infty) + F(t), \end{aligned} \tag{3.13}$$

where A, x_0, E are constants, $Z(x)$ is an arbitrary sufficiently smooth function, and $F(t)$ satisfies

$$F'(t) + \lambda D_t^\beta F(t) - Q_h F(t) = 0. \tag{3.14}$$

Here D_t^β denotes the fractional derivative appearing in the governing system. Then the corresponding infinitesimal generators are

$$\begin{aligned} \boldsymbol{X}_1 &= \frac{\partial}{\partial x}, \\ \boldsymbol{X}_2 &= x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \\ \boldsymbol{X}_3[Z] &= Z(x)\frac{\partial}{\partial y} + Z'(x)u\frac{\partial}{\partial v}, \\ \boldsymbol{X}_4 &= (T - T_\infty)\frac{\partial}{\partial T}, \\ \boldsymbol{X}_5[F] &= F(t)\frac{\partial}{\partial T}. \end{aligned} \tag{3.15}$$

These vector fields form a Lie algebra under the usual commutator bracket. The family $\{\mathbf{X}_3[Z]\}$ is an infinite-dimensional abelian ideal, provided the admissible class of functions Z is closed under differentiation and the operation $Z \mapsto xZ'$. The thermal generators $\mathbf{X}_4$ and $\mathbf{X}_5[F]$ form a Lie subalgebra. Moreover, $\xi^1 = 0$ implies that the reported infinitesimal symmetries have no ∂_t component.

Proof. We start from the general infinitesimal generator (2.6). The infinitesimal invariance criterion requires the prolonged generator to annihilate each equation of the governing system on its solution manifold. For the time-fractional system, the fractional prolongation is computed consistently with the definition of $\mathcal{D}_t^\beta$ adopted in the model. Throughout the calculation, we assume $\lambda \neq 0$ and $\nu \neq \kappa$, together with the regularity conditions required for the fractional prolongation and the splitting of the determining equations.

Substituting the prolonged generator into the invariance criterion and splitting with respect to the independent derivative variables and the functionally independent terms in u, v, and T yields the reduced determining equations. The determining equations associated with the time-fractional terms imply

$$\xi^1 = 0. \tag{3.16}$$

The spatial determining equations give

$$\frac{\partial \xi^2}{\partial x} = A, \qquad \frac{\partial \xi^2}{\partial t} = \frac{\partial \xi^2}{\partial y} = \frac{\partial \xi^2}{\partial u} = \frac{\partial \xi^2}{\partial v} = \frac{\partial \xi^2}{\partial T} = 0, \tag{3.17}$$

where A is a constant. Integrating (3.17) with respect to x, we obtain

$$\xi^2 = Ax + x_0, \tag{3.18}$$

where x_0 is an arbitrary constant. The remaining spatial determining equations yield

$$\xi^3 = Z(x), \qquad \eta^u = Au, \qquad \eta^v = Z'(x)u, \tag{3.19}$$

where Z is an admissible differentiable function. The assumptions on the parameters eliminate the additional coefficients arising from the spatial determining equations, and the remaining spatial relations are satisfied identically after (3.18) and (3.19) are imposed. The thermal determining equations imply

$$\eta^T = E(T - T_\infty) + F(t), \tag{3.20}$$

where E is a constant and F depends only on t. In the notation of the determining equations, the fractional contribution associated with F is $B_2 = \mathcal{D}_t^\beta F$. The remaining thermal determining equation is

$$F'(t) + \lambda \mathcal{D}_t^\beta F(t) - Q_h F(t) = Q_h T_\infty E. \tag{3.21}$$

Consequently, the reduced determining system is

$$\begin{aligned} &\xi^1 = 0, \\ &\xi^2 = Ax + x_0, \qquad \xi^3 = Z(x), \\ &\eta^u = Au, \qquad \eta^v = Z'(x)u, \\ &\eta^T = E(T - T_\infty) + F(t), \qquad \textit{where} \quad F' + \lambda \mathcal{D}_t^\beta F - Q_h F = Q_h T_\infty E \end{aligned} \tag{3.22}$$

Substituting the obtained infinitesimal coefficients into the reduced invariance conditions, we obtain the residual expressions $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, given by

$$\begin{aligned} \mathcal{C}_1 &\equiv 0, \\ \mathcal{C}_2 &= A\Delta_2, \\ \mathcal{C}_3 &= E\Delta_3 + F' + \lambda \mathcal{D}_t^\beta F - Q_h F - Q_h T_\infty E. \end{aligned} \tag{3.23}$$

Here, Δ_2 and Δ_3 denote the corresponding equations of the governing system, in the normalization used to define the residuals. Thus, on the solution manifold, all invariance conditions are satisfied precisely when (3.21) holds.

Since $\mathcal{D}_t^\beta$ annihilates constants, the inhomogeneous memory equation can be reduced to a homogeneous one by writing

$$F(t) = -T_\infty E + F_0(t). \tag{3.24}$$

Substitution into (3.21) gives

$$F_0'(t) + \lambda \mathcal{D}_t^\beta F_0(t) - Q_h F_0(t) = 0. \tag{3.25}$$

The admissible functions F_0 are determined by this fractional equation and the initial-data requirements of the chosen derivative. In particular, the appropriate initial conditions depend on the definition of $\mathcal{D}_t^\beta$; they should not be specified solely from the order of the derivative without fixing that definition.

For the Lie-algebra calculation, it is convenient to use (3.24) to rewrite the thermal coefficient as

$$\eta^T = ET + F_0(t). \tag{3.26}$$

The infinitesimal generator therefore becomes

$$\boldsymbol{X} = x_0 \frac{\partial}{\partial x} + A\left(x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\right) + Z(x)\frac{\partial}{\partial y} + Z'(x)u\frac{\partial}{\partial v} + ET\frac{\partial}{\partial T} + F_0(t)\frac{\partial}{\partial T}. \tag{3.27}$$

This representation is equivalent to the original thermal coefficient, provided that F and F_0 are related by (3.24). Separating the constant parameters and the function-valued parameters gives the generators in (3.15), with the thermal scaling generator taken as

$$\boldsymbol{X}_4 = T\frac{\partial}{\partial T} \tag{3.28}$$

and the homogeneous thermal generators defined by

$$\boldsymbol{X}_5[F_0] = F_0(t)\frac{\partial}{\partial T}, \qquad F_0{}' + \lambda \mathcal{D}_t^{\beta} F_0 - Q_h F_0 = 0. \tag{3.29}$$

We now calculate the commutators. Define

$$\boldsymbol{X}_1 = \frac{\partial}{\partial x}, \qquad \boldsymbol{X}_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \qquad \boldsymbol{X}_3[Z] = Z(x)\frac{\partial}{\partial y} + Z'(x)u\frac{\partial}{\partial v}. \tag{3.30}$$

Direct computation gives

$$[\boldsymbol{X}_1, \boldsymbol{X}_2] = \boldsymbol{X}_1, \qquad \left[\boldsymbol{X}_1, \boldsymbol{X}_3[Z]\right] = \boldsymbol{X}_3[Z'], \qquad \left[\boldsymbol{X}_2, \boldsymbol{X}_3[Z]\right] = \boldsymbol{X}_3[xZ']. \tag{3.31}$$

For any two admissible functions Z_1 and Z_2, we have

$$\left[\boldsymbol{X}_3[Z_1], \boldsymbol{X}_3[Z_2]\right] = 0. \tag{3.32}$$

Hence the family $\{\mathbf{X}_3[Z]\}$ is abelian. It is an ideal in the Lie algebra generated by $\mathbf{X}_1$, $\mathbf{X}_2$, and the fields $\mathbf{X}_3[Z]$, provided the admissible class of functions Z is closed under differentiation and multiplication by x.

For the thermal generators (3.28) and (3.29), direct computation yields

$$\left[\boldsymbol{X}_4, \boldsymbol{X}_5[F_0]\right] = -\boldsymbol{X}_5[F_0], \qquad \left[\boldsymbol{X}_5[F_0], \boldsymbol{X}_5[G_0]\right] = 0, \tag{3.33}$$

where F_0 and G_0 satisfy (3.25). All commutators between the thermal generators and the spatial generators vanish. The thermal family is therefore closed under the Lie bracket, provided that the admissible solution space of (3.25) is a vector space. The full displayed family is closed under the Lie bracket when the spatial and thermal admissible classes satisfy the stated closure conditions.

Finally, (3.16) shows that the infinitesimal generator has no component in the time direction. The determining equations therefore yield the stated spatial and thermal symmetry family, subject to the specified fractional prolongation and the thermal memory condition. □

4. Symmetry-Generated Exact Solutions and Their Graphical Analysis

We investigate representative exact solutions associated with the one-parameter transformations generated by the infinitesimal *operators* $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{X}_3[Z]$, $\boldsymbol{X}_4$ *and* $\boldsymbol{X}_5[F]$ defined in (3.15). To capture the spatial profiles, scaling properties, multidimensional dependencies, and comparative behaviors across parameter spaces, our analysis integrates both detailed two-dimensional (2D) comparative curves and three-dimensional (3D) surface and contour representations.

For the exact-solution analysis, we assume that the radiative heat-flux contribution vanishes ($q_r = 0$; noting that identical conclusions hold if q_r is a linear function of T_y, such as a linearized Rosseland flux), and we denote

$$a_T = \frac{k}{\rho c_p}, \qquad M = \frac{\sigma B_0^2}{\rho}, \qquad Q = \frac{Q_0}{\rho c_p}.$$

We further stipulate the physical constraints

$$\nu > 0, \qquad a_T > 0, \qquad M > 0, \qquad \beta > 0, \qquad Q = -a_T\beta^2,$$

ensuring that the thermal source term acts strictly as an internal heat sink ($Q_0 < 0$). Under these assumptions, the governing system (2.2) admits the fundamental stationary solution

$$\begin{aligned} u_0(x,y,t) = U_0 e^{-\mu y}, \qquad v_0(x,y,t) = 0, \qquad T_0(x,y,t) \\ = T_\infty + \Theta_0 e^{-\beta y}, \end{aligned} \tag{4.1}$$

where

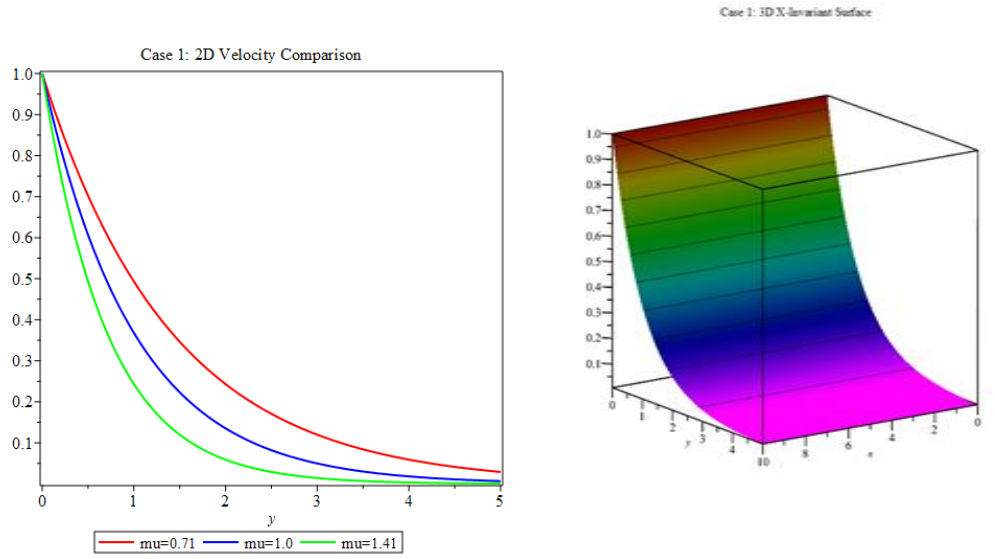
$$\mu = \sqrt{\frac{M}{\nu}},$$

and U_0, Θ_0 are constant amplitudes.

Under the aforementioned conditions, each infinitesimal generator in (3.15) maps solutions of (2.2) to new solutions. For $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, and $\boldsymbol{X}_3[Z]$, this mapping is confirmed via direct substitution of the finite transformations. For $\mathbf{X}_4$, the thermal equation must remain linear and homogeneous in $T - T_\infty$, whereas for $\boldsymbol{X}_5[F]$, the auxiliary function $F(t)$ must satisfy the governing memory equation (4.4).

4.1 Case 1: Spatial translation generated by $\mathbf{X_1} = \boldsymbol{\partial_x}$.

The generator $\mathbf{X}_1 = \partial/\partial x$ induces the spatial translation $x^* = x + \varepsilon$. The invariant-surface conditions impose $u_x = 0$, $v_x = 0$, and $T_x = 0$. Because the base solution (4.1) is inherently independent of x, the translation leaves it invariant such that $u_\varepsilon = u_0$, $v_\varepsilon = v_0$, and $T_\varepsilon = T_0$. To provide a comprehensive comparative perspective, Figure 1 displays both 2D transverse profiles for varying magnetic parameter μ alongside a 3D extrusion demonstrating complete longitudinal x-invariance. The 2D comparative curves clearly illustrate how an increase in μ steepens the momentum boundary layer, whereas variations in β independently dictate the thermal boundary layer thickness.



(a) 2D velocity comparison for various μ. **(b) 3D surface showing x-invariance.**

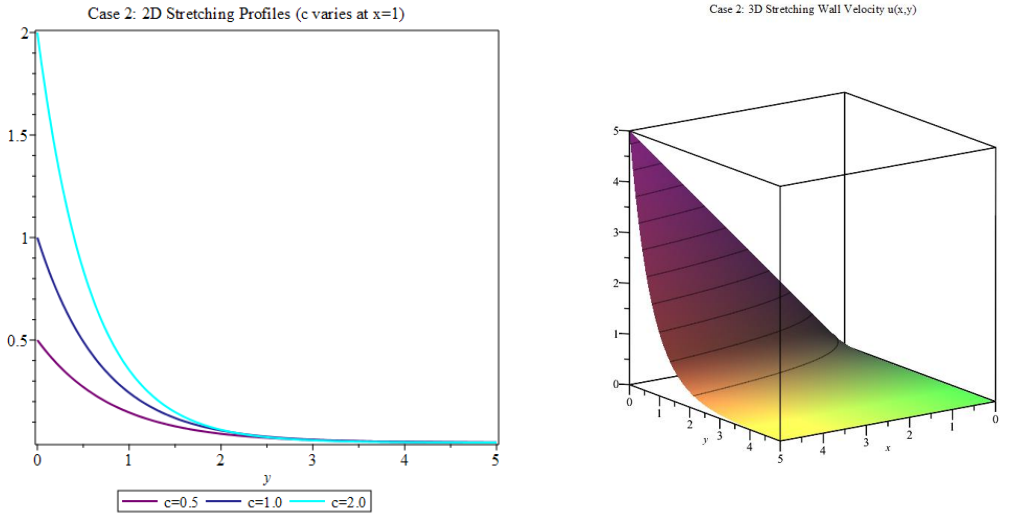
Figure 1: Transverse profiles and 3D spatial structure for the translation-invariant solution.

4.2 Case 2: Scaling generated by $X_2 = x\partial_x + u\partial_u$.

Considering the scaling generator $X_2 = x\partial_x + u\partial_u$, its finite transformations yield $x^* = e^{\varepsilon}x$ and $u^* = e^{\varepsilon}u$. Applying this operator to (4.1) generates the exact family $u_\varepsilon = U_0 e^{\varepsilon} e^{-\mu y}$. Furthermore, the reduced system yields the stretching-wall invariant solution:

$$u = c\,x\,e^{-\kappa y}, \qquad v = -\frac{c}{\kappa}(1 - e^{-\kappa y}), \qquad \kappa = \sqrt{\frac{c+M}{\nu}}, \tag{4.2}$$

subject to $c + M > 0$. Figure 2 contrasts the 2D velocity profiles across various stretching rates c with the scaling parameter ε, accompanied by a 3D isometric surface that highlights linear spatial growth along x coupled with exponential decay along y.

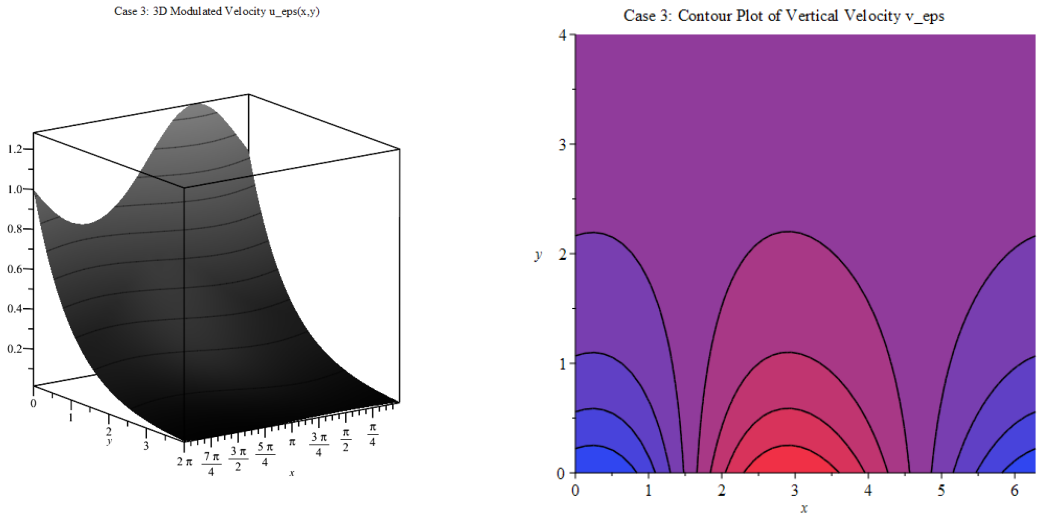


(a) 2D velocity profiles for varying c. **(b) 3D stretching-wall velocity surface $u(x, y)$.**

Figure 2: Comparative 2D curves and 3D surface for the X_2-generated stretching-wall solution.

4.3 Case 3: Spatial modulation generated by $\mathbf{X}_3[Z]$.

Letting $\mathbf{X}_3[Z] = Z(x)\,\partial_y + Z'(x)u\,\partial_v$, this operator generates the coordinate shift $y^* = y + \varepsilon Z(x)$ and $v^* = v + \varepsilon Z'(x)u$. For graphical demonstration, we select $Z(x) = \sin x$ and $\varepsilon = 0.25$. Figure 3 presents a multi-view 3D surface and contour comparison illustrating how spatial modulation alters both velocity components and the temperature excess. Notably, the vertical velocity component $v_\varepsilon \propto \cos x$ exhibits periodic sign reversals, driving fluid alternately toward and away from the perturbed wavy boundary.



(a) 3D velocity $u_\varepsilon(x, y)$. **(b) Contour plot of vertical velocity v_ε.**

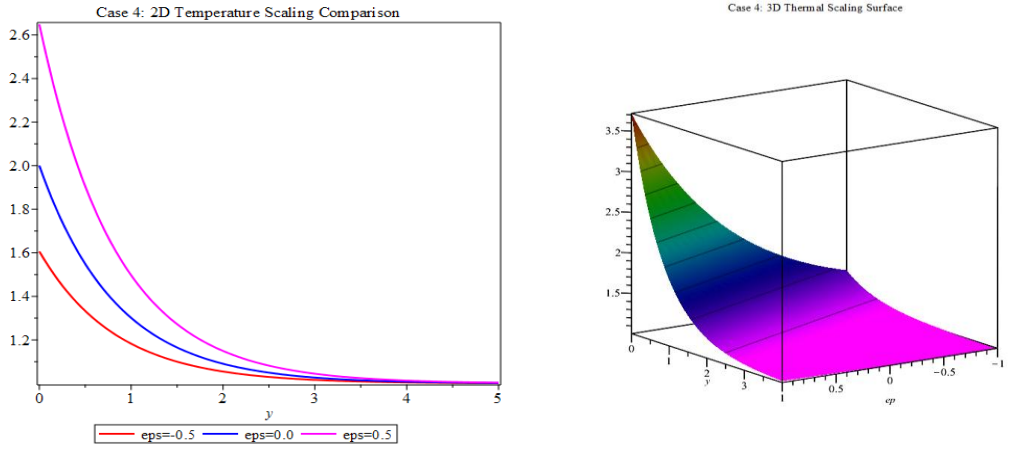
Figure 3: Three-dimensional surface and contour comparison induced by $\mathbf{X}_3[\sin x]$ ($\varepsilon = 0.25$).

4.4 Case 4: Temperature scaling generated by $\mathbf{X}_4$.

The temperature-scaling generator $\mathbf{X}_4 = (T - T_\infty)\,\partial_T$ produces the transformed temperature field

$$T_\varepsilon(y) = T_\infty + e^{\varepsilon}\Theta_0 e^{-\beta y}. \tag{4.3}$$

Figure 4 contrasts 2D profiles across multiple scaling parameters ε alongside a 3D parametric surface that visualizes thermal amplitude modulation across the transverse and scaling domains.



(a) 2D temperature excess for various ε. **(b) 3D thermal scaling surface.**

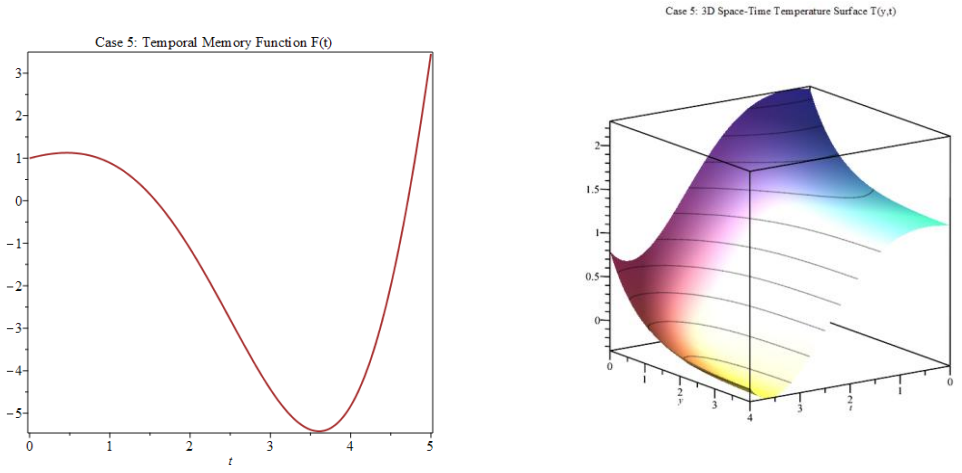
Figure 4: Comparative 2D and 3D temperature profiles generated by $\mathbf{X}_4$.

4.5 Case 5: Additive thermal transformations generated by $\mathbf{X}_5[F]$.

Considering $\mathbf{X}_5[F] = F(t)\,\partial_T$, where $F(t)$ satisfies the fractional memory equation

$$F'(t) + \lambda\, {}^{C}\mathrm{D}_t^{\alpha} F'(t) - QF(t) = 0, \quad 0 < \alpha < 1, \tag{4.4}$$

the resulting exact solution family is given by $T_\varepsilon(y,t) = T_\infty + \Theta_0 e^{-\beta y} + \varepsilon F(t)$. Figure 5 illustrates the time-dependent temperature shift through a 2D numerical profile of $F(t)$ and a 3D space-time-temperature surface $T_\varepsilon(y,t)$ that captures the combined spatial decay and fractional temporal memory evolution.



(a) 2D temporal evolution $F(t)$ for various α. **(b) 3D space-time temperature surface $T_\varepsilon(y,t)$.**

Figure 5: Fractional temporal memory function and 3D space-time thermal evolution for Case 5.

The expanded 2D and 3D visualizations clearly delineate the distinct structural roles of each symmetry group: translational invariance in Case 1, stretching-wall dynamics and amplitude scaling in Case 2, multidimensional spatial modulation in Case 3, thermal excess rescaling in Case 4, and fractional memory-dependent temporal evolution in Case 5.

5. Conclusions

In this study, the Lie point symmetries and exact solutions of the time-fractional magnetohydrodynamic (MHD) boundary layer flow and heat transfer system for a viscoelastic fluid over a stretching sheet were comprehensively analyzed. By employing the Gazizov–Kasatkin–Lukashchuk (GKL) fractional prolongation framework and appropriate standing assumptions, the overdetermined determining equations were resolved to establish the full set of infinitesimal generators. The resulting symmetry group structure comprises spatial translation, scaling transformations, an infinite-dimensional abelian ideal of spatial modulations, temperature scaling, and a family of additive thermal transformations governed by a linear fractional memory equation. Through detailed exact-solution analysis and advanced graphical visualization, including 2D comparative curves and 3D surface and contour plots, the physical impacts of key parameters were explicitly demonstrated. Specifically, variations in the magnetic parameter (μ) and fractional order (α) significantly alter the momentum and thermal boundary layer thicknesses, whereas spatial modulation

generators ($\mathbf{X}_3[Z]$) induce periodic transverse velocity reversals that direct fluid alternately toward and away from a wavy boundary. Furthermore, thermal scaling ($\mathbf{X}_4$) and fractional memory-dependent temporal generators ($\mathbf{X}_5[F]$) accurately capture amplitude modulation alongside the coupled evolution of spatial decay with temporal memory. These findings provide deeper analytical insights into non-Newtonian fluid dynamics with memory effects and offer benchmark exact solutions for validating numerical schemes in fractional differential systems**.**

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