Spectral Collocation with Variable Transformation for Riemann–Liouville Fractional Integral Equations with Non-Smooth Solutions: Stability and Conditioning Analysis

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ABSTRACT

Weakly singular Volterra integral equations of Riemann–Liouville type with rational order have non-smooth solutions, so polynomial collocation converges slowly. This paper analyses spectral collocation after a power-law change of the independent variable that makes the solution smooth. The scheme is the Chebyshev–Gauss case of the Müntz collocation method of D. Hou, Y. Lin, M. Azaiez and C. Xu (J. Sci. Comput. 81 (2019) 2162–2187). The main result is an explicit bound on the condition number that grows like the squared logarithm of the polynomial degree. Its constant uses a new resolvent estimate based on the complete monotonicity of Mittag-Leffler functions. For dissipative problems this removes the exponential growth of the classical Gronwall-type constant: the bound grows quadratically in the coefficient, as observed, and is 35 to 53 times the measured value. For kernels bounded below by a positive constant, a matching lower bound shows that the exponential growth is genuine. Stability for every rational order, error bounds with one logarithmic factor and analyticity of the transformed solution are also proved under Hölder continuity of the kernel; all results are asymptotic. On five test problems, including a fractional relaxation model, the scheme reaches errors of 10^{-13} to 10^{-15} with at most 64 unknowns in 0.4 to 1.8 ms. The method of T. Pu and M. Fasondini (Adv. Comput. Math. 49 (2023) 7), with high-precision matrices, is up to two orders of magnitude more accurate, whereas collocation runs in double precision, accepts non-separable kernels and was several hundred times faster.

1. Introduction

We consider the second-kind Volterra equation

$$c(x)\,y(x) = g(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\,K(x,s)\,y(s)\,ds, \qquad x \in [0,1], \tag{1.1}$$

with $0 < \alpha < 1$, $c \geq c_0 > 0$ and a continuous kernel K. For $K(x,s) = h(x)$ the integral is $h(x)(I^{\alpha}y)(x)$ with the Riemann–Liouville integral I^{α}. Equations of this form arise from Caputo fractional initial value problems and from viscoelastic and dielectric relaxation [1,2]. Even for smooth data the solution contains powers $x^{j+k\alpha}$, and y' is unbounded at $x = 0$ [3, Ch. 6]. Polynomial approximation in x therefore converges only at the rate $N^{-2\alpha}$.

Related work

Graded meshes were the first remedy. Vainikko and Uba [4] and Brunner [5] obtained high-order piecewise polynomial collocation. Lubich introduced fractional multistep methods [6]. Diogo, McKee and Tang transformed the unknown [7]. Pedas and Vainikko introduced smoothing transformations of the independent variable [8]. Global spectral methods followed. Chen and Tang treated weakly

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singular kernels with smooth solutions for $\mu = 1 - \alpha < 1/2$ [9], and non-smooth solutions by function and variable transformations [10]. Li and Tang treated $\mu = 1/2$ with $t = z^2$ [11]. Other methods use singular bases: poly-fractonomials [12], generalized Jacobi functions [13] and sum spaces [14]. Hou and Xu introduced fractional Jacobi polynomials [15]. Hou, Lin, Azaiez and Xu then analysed Müntz collocation for weakly singular Volterra equations [16]. With $\lambda = 1/q$ and Chebyshev parameters, their scheme is the one studied here. Smoothing transformations were also used for integro-differential problems [17], and hp versions exist [18]. Finally, Pu and Fasondini used Jacobi polynomials in fractional powers with exact integration matrices computed in high precision (the JFP method) [19].

What is missing

In [9–11,16] the error bounds depend on derivatives of a transformed kernel, through quantities such as K^* in [16], and the stability constants are not explicit. None of [9–11,16,19] bounds the condition number of the linear system, although [19] reports it numerically. For large coefficients, the Gronwall-type constant $E_\alpha(\|k\|_\infty)$ grows like $\exp\left(\|k\|^{1/\alpha}\right)$ and makes all such bounds uninformative.

Main result

Let Λ_N be the Lebesgue constant of Chebyshev–Gauss interpolation, $k = K/c$, and $c_q = \Gamma(q+1)/\Gamma(q+\alpha)$.

Theorem A. Let $\alpha = p/q$ and let k be Hölder continuous in x of some order $\beta \in (0,1]$. For $N \geq N_0$ the collocation matrix B_N satisfies

$$\kappa_\infty(B_N) \leq 2\Gamma_k \Lambda_N\left(1 + c_q\|k\|_\infty \Lambda_N\right) = O\left(\log^2 N\right), \qquad \Gamma_k \leq \min\left\{E_\alpha(\|k\|_\infty),\ \inf_{\sigma \geq 0} 2E_\alpha(2\|k+\sigma\|_\infty)\right\}.$$

For $k \equiv -\lambda^2$ the second bound gives $\Gamma_k \leq 2$ and $\kappa_\infty = O\left(\lambda^2 \log^2 N\right)$. This matches the observed λ^2 growth. With $\Gamma_k = E_{1/2}\left(\lambda^2\right)$ instead, the bound exceeds 10^{270} at $\lambda = 5$ (Section 6.2). The bound has the right growth rate but a pessimistic constant. For kernels $k \geq \kappa_0 > 0$ no such gain is possible, since $\Gamma_k \geq E_\alpha(\kappa_0)$ (Proposition 3.3). As in [9–11,16], Theorem A is asymptotic. It holds for $N \geq N_0$, and N_0 is of order 10^5–10^7 for our test problems (Remark 5.2). Theorem A follows from Theorems 3.1 and 5.6. We also prove stability for every $\alpha = p/q \in (0,1)$ (Theorem 5.3). We prove error bounds with one $\log N$ factor (Theorem 5.4). They need only Hölder continuity of k in x and the approximability of one fixed function. Finally, we prove that the transformed solution is analytic for analytic data (Theorem 4.2). We compare the scheme with the JFP method and with direct collocation, and apply it to a fractional relaxation model.

2. The transformed equation

Throughout, $\alpha = p/q$ with $p < q$ coprime. The maximum norm on $[0,1]$, or on the triangle $\Delta = \{0 \leq s \leq x \leq 1\}$, is $\|\cdot\|$. $\mathbb{P}_N$ is the space of polynomials of degree at most N, and $E_N(w) = \min_{r \in \mathbb{P}_N}\|w - r\|$. We use $E_{\alpha,\beta}(z) = \sum_{j \geq 0} z^j/\Gamma(\alpha j + \beta)$, $E_\alpha = E_{\alpha,1}$, and $S_q(v) = 1 + v + \cdots + v^{q-1}$.

Dividing (1.1) by c gives $y = f + \mathcal{K}y$ with $f = g/c$ and kernel $k = K/c$. We assume:

(A1) $k \in C(\Delta)$;

(A2) $|k|_\beta := \sup_s \sup_{s \le x_1 < x_2 \le 1} |k(x_2, s) - k(x_1, s)|/(x_2 - x_1)^\beta < \infty$ for some $\beta \in (0,1]$.

We write $\kappa = \min(\alpha, \beta)$. (A2) holds with $\beta = 1$ for every kernel that is Lipschitz in x. With $x = t^q$ set $Y(t) = y(t^q)$, $F(t) = f(t^q)$, $\widetilde{k}(t,\tau) = k(t^q, \tau^q)$ and $(\mathcal{K}_\alpha \varphi)(t) = (\mathcal{K}\Phi)(t^q)$, $\Phi(x) = \varphi\left(x^{1/q}\right)$. Then (1.1) becomes

$$Y = F + \mathcal{K}_\alpha Y \qquad \text{on } [0,1], \tag{2.1}$$

and maximum-norm errors for Y and y coincide.

Lemma 2.1.

(i) (Dilation form [16].) For $\varphi \in C[0,1]$,

$$(\mathcal{K}_\alpha \varphi)(t) = \frac{q\, t^p}{\Gamma(\alpha)} \int_0^1 (1-v)^{\alpha-1}\, v^{q-1}\, G_t(v)\, \varphi(tv)\, dv, \qquad G_t(v) = S_q(v)^{\alpha-1} \widetilde{k}(t, tv). \tag{2.2}$$

(ii) If $k \equiv 1$, then $\mathcal{K}_\alpha t^n = \gamma_n t^{n+p}$ with $\gamma_n = \Gamma(n/q+1)/\Gamma(n/q+\alpha+1)$, which is positive and decreasing in $n \ge 0$.

(iii) $\left\|\mathcal{K}_\alpha^j \varphi\right\| \le \|k\|^j \|\varphi\|/\Gamma(j\alpha+1)$, hence $\left\|(I - \mathcal{K}_\alpha)^{-1}\right\| \le E_\alpha(\|k\|)$.

(iv) $|\mathcal{K}_\alpha \varphi|_\kappa \le L_k \|\varphi\|$ with $L_k = q^\kappa\left(2\|k\| + |k|_\beta\right)/\Gamma(\alpha+1)$.

Proof. (i) Put $s = xu$ and then $x = t^q$, $u = v^q$; use $1 - v^q = (1-v)S_q(v)$. (ii) is the power rule for I^α; γ_n decreases because the digamma function is increasing. (iii) $\left|\mathcal{K}^j \Phi\right| \le \|k\|^j I^{j\alpha} |\Phi|$ by positivity of I^α and the semigroup law. (iv) For $x_1 < x_2$ split $\Gamma(\alpha)\left(\mathcal{K}\Phi(x_2) - \mathcal{K}\Phi(x_1)\right)$ into $\int_0^{x_1}\left[(x_2 - s)^{\alpha-1} - (x_1 - s)^{\alpha-1}\right] k(x_2, s)\Phi\, ds$, $\int_0^{x_1} (x_1 - s)^{\alpha-1}\, [k(x_2, s) - k(x_1, s)]\Phi\, ds$ and $\int_{x_1}^{x_2} (x_2 - s)^{\alpha-1}\, k(x_2, s)\Phi\, ds$, bounded by $\|k\|\|\Phi\|(x_2 - x_1)^\alpha/\alpha$, $|k|_\beta \|\Phi\|(x_2 - x_1)^\beta/\alpha$ and $\|k\|\|\Phi\|(x_2 - x_1)^\alpha/\alpha$; since $x_2 - x_1 \le 1$, each is at most a multiple of $(x_2 - x_1)^\kappa$. Finally $t_2^q - t_1^q \le q(t_2 - t_1)$. □

3. Resolvent bounds

Let $\Gamma_k = \left\|(I - \mathcal{K}_\alpha)^{-1}\right\|$; by conjugation with $x = t^q$ it equals $\left\|(I - \mathcal{K})^{-1}\right\|$ on $C[0,1]$. Lemma 2.1(iii) gives $\Gamma_k \le E_\alpha(\|k\|)$, which grows like $\exp\left(\|k\|^{1/\alpha}\right)/\alpha$. The next result removes this growth for kernels that are close to a non-positive constant.

Theorem 3.1. Under (A1), for every $\sigma \ge 0$,

$$\Gamma_k \le 2\, E_\alpha(2\|k + \sigma\|).$$

Proof. Write $k = -\sigma + d$. The equation $y + \sigma I^\alpha y = w$ has the solution $y = R_\sigma w := w - r_\sigma * w$, with $r_\sigma(u) = \sigma u^{\alpha-1} E_{\alpha,\alpha}(-\sigma u^\alpha)$ [20]. For $0 < \alpha \le 1$, $E_{\alpha,\alpha}(-z)$ is completely monotone [20], so $r_\sigma \ge 0$, and $\int_0^x r_\sigma = 1 - E_\alpha(-\sigma x^\alpha) \in [0,1]$ by [21]. Hence, with $m_w(x) = \max_{[0,x]} |w|$, we have $|R_\sigma w|(x) \le 2m_w(x)$.

The equation $y = f + \mathcal{K}y$ is equivalent to $y = R_\sigma f + R_\sigma \mathcal{D} y$, where $\mathcal{D}$ has kernel d. For $w \in C[0,1]$, $|\mathcal{D}w| \le \|d\| I^\alpha m_w$, and $I^\alpha m_w$ is non-decreasing because m_w is. Therefore $m_{R_\sigma \mathcal{D} w} \le 2\|d\|\, I^\alpha m_w$, and by induction $m_{(R_\sigma \mathcal{D})^j w} \le (2\|d\|)^j I^{j\alpha} m_w \le (2\|d\|)^j \|w\|/\Gamma(j\alpha+1)$. Summing the Neumann series gives $\left\|(I - R_\sigma \mathcal{D})^{-1}\right\| \le E_\alpha(2\|d\|)$, and $\|y\| \le E_\alpha(2\|d\|) \cdot 2\|f\|$. □

Remark 3.2. For $k \equiv -\lambda^2$, taking $\sigma = \lambda^2$ gives $\Gamma_k \le 2$ for every λ. Since $r_\sigma \ge 0$, in fact $\Gamma_k = \left\|R_{\lambda^2}\right\| = 1 + \int_0^1 r_{\lambda^2} = 2 - E_\alpha\left(-\lambda^2\right)$, so the bound is sharp as $\lambda \to \infty$. For general kernels, $\sigma \mapsto \|k + \sigma\| = \max(\max k + \sigma, -\min k - \sigma)$ is convex and piecewise linear with minimum at $\sigma^* = -(\max k + \min k)/2$; under the constraint $\sigma \ge 0$ the best choice is $\max(\sigma^*, 0)$.

Proposition 3.3. If $k \ge \kappa_0 \ge 0$ on Δ, then $E_\alpha(\kappa_0) \le \Gamma_k \le E_\alpha(\|k\|)$. In particular $\Gamma_k = E_\alpha\left(\lambda^2\right)$ for $k \equiv \lambda^2$.

Proof. The upper bound is Lemma 2.1(iii). Since $k \ge \kappa_0 \ge 0$, every term of the Neumann series for $f \equiv 1$ is non-negative and $\mathcal{K}^j 1 \ge \kappa_0^j I^{j\alpha} 1 = \kappa_0^j x^{j\alpha}/\Gamma(j\alpha+1)$. At $x = 1$ the sum is $E_\alpha(\kappa_0)$, so $\left\|(I - \mathcal{K})^{-1} 1\right\| \ge E_\alpha(\kappa_0)$. □

So the exponential constant reflects growing solutions, not a weak analysis. Theorem 3.1 improves the constant exactly in the dissipative case. Take $k \equiv \lambda^2$ with $\lambda = 1, 1.5, 2$ and $\alpha = 1/2$. The measured $\left\|B_{64}^{-1}\right\|_\infty$ is 5.007, 315.3 and 1.769×10^7. The exact values $E_{1/2}\left(\lambda^2\right)$ are 5.009, 315.7 and 1.777×10^7.

4. Regularity

Theorem 4.1. If $f \in C^m[0,1]$ and $k \in C^m(\Delta)$, $m \ge 1$, then $Y \in C^m[0,1]$.

Proof. By [3, Ch. 6] (see also [10, Lemma 1.1]), $y(x) = \sum_{j+k\alpha<m} \gamma_{jk}\, x^{j+k\alpha} + y_m(x)$ with $y_m \in C^m[0,1]$. Since $q(j + k\alpha) \in \mathbb{N}$, the substitution $x = t^q$ maps each term to a monomial or to a C^m function. □

For $R \ge 1$ let W_R ($W_R^{(2)}$) be the Banach algebra of power series in one (two) variables with $\|\sum u_n t^n\|_R = \sum |u_n| R^n$ ($\left\|\sum k_{ij} t^i \tau^j\right\|_R = \sum |k_{ij}| R^{i+j}$).

Theorem 4.2. If $F \in W_R$ and $\widetilde{k} \in W_R^{(2)}$ for some $R \ge 1$, then $Y \in W_R$ and

$$\|Y\|_R \le \|F\|_R\, E_\alpha\left(R^p \left\|\widetilde{k}\right\|_R\right).$$

If $R > 1$, Y is analytic and bounded in the Bernstein ellipse $\mathcal{E}_\rho$ of [0,1] whenever $\rho + \rho^{-1} < 4R - 2$.

Proof. For a power series u, $\mathcal{K}_\alpha u = \sum_{i,j} k_{ij}\, t^i\, \mathcal{K}_\alpha^{(1)}(\tau^j u)$, where $\mathcal{K}_\alpha^{(1)}$ is the operator with $k \equiv 1$. If u contains only powers t^n with $n \ge jp$, Lemma 2.1(ii) gives $\|\mathcal{K}_\alpha u\|_R \le \gamma_{jp} R^p \left\|\widetilde{k}\right\|_R \|u\|_R$, and $\mathcal{K}_\alpha u$ contains only powers $n \ge (j+1)p$. By induction $\left\|\mathcal{K}_\alpha^j F\right\|_R \le \left(R^p \left\|\widetilde{k}\right\|_R\right)^j \|F\|_R \prod_{i<j} \gamma_{ip}$, and since

$ip/q = i\alpha$ the product telescopes to $1/\Gamma(j\alpha + 1)$. The Neumann series converges in W_R, hence uniformly on [0,1], to the unique solution. The ellipse $\mathcal{E}_\rho$ has centre $1/2$ and semi-axis $(\rho + \rho^{-1})/4$. □

5. The scheme and its analysis

Let $t_s = \frac{1}{2}\left(1 - \cos\frac{(2s+1)\pi}{2N+2}\right)$, $s = 0, \dots, N$. Let I_N be interpolation at these points, with Lagrange basis ℓ_j. Recall [22] that

$$\Lambda_N = \|I_N\| \le 2/\pi \log(N+1) + 1. \tag{5.1}$$

Let $(v_k, w_k)_{k=1}^M$ be the Gauss–Jacobi rule on [0,1] for the weight $(1-v)^{\alpha-1}v^{q-1}$ [23]. Its weights are positive and sum to $B(\alpha, q)$. Replacing the integral in (2.2) by this rule defines $\mathcal{K}_\alpha^M$. The scheme seeks $Y_N \in \mathbb{P}_N$ with

$$Y_N(t_s) = F(t_s) + \left(\mathcal{K}_\alpha^M Y_N\right)(t_s), \qquad s = 0, \dots, N, \tag{5.2}$$

that is, $B_N \mathbf{Y} = \mathbf{F}$ with

$$B_N = I - Q, \qquad Q_{sj} = \frac{q\, t_s^p}{\Gamma(\alpha)} \sum_{k=1}^{M} w_k \; G_{t_s}(v_k)\, \ell_j(t_s v_k). \tag{5.3}$$

With $M = N + 1$ this is the scheme of [16] with collocation at the zeros of the fractional Jacobi polynomial $J_{N+1}^{-1/2,-1/2,1/q}$. We evaluate $\ell_j(t_s v_k)$ by the barycentric formula [24]. Assembly costs $O(N^2 M)$ operations and the solve $O(N^3)$.

Let $c_q = qB(\alpha, q)/\Gamma(\alpha) = \Gamma(q+1)/\Gamma(q+\alpha)$, $L = 2M - 1 - N$, and $\varepsilon_L = \sup_{t\in[0,1]} E_L(G_t)$.

Lemma 5.1. For $\varphi \in \mathbb{P}_N$, $\left\|\mathcal{K}_\alpha^M \varphi\right\| \le c_q \|k\| \|\varphi\|$ and $\left\|\left(\mathcal{K}_\alpha - \mathcal{K}_\alpha^M\right)\varphi\right\| \le 2c_q \varepsilon_L \|\varphi\|$. If $\tilde{k}$ is analytic in a complex neighbourhood of $\{(t, tv) : t, v \in [0,1]\}$, then $\varepsilon_L \le C\rho_k^{-L}$ for some $\rho_k > 1$.

Proof. Let $r_t \in \mathbb{P}_L$ attain $E_L(G_t)$. The function $v \mapsto r_t(v)\varphi(tv)$ has degree at most $2M - 1$ and is integrated exactly, and $|(G_t - r_t)(v)\varphi(tv)| \le \varepsilon_L \|\varphi\|$. Both the integral and the rule have positive weights of mass $B(\alpha, q)$, and $0 < S_q^{\alpha-1} \le 1$ on [0,1]. The analytic case follows because $S_q^{\alpha-1}$ is analytic near [0,1] (its singularities are the q-th roots of unity other than 1) and from Bernstein's theorem [25]. □

Jackson's theorem [22] gives $E_N(F) \le 6\,\omega(F, 1/N)$ on $[-1,1]$. Mapping [0,1] affinely onto $[-1,1]$ halves the modulus of continuity, so $E_N(w) \le C_J N^{-\kappa} |w|_\kappa$ on [0,1] with $C_J = 6 \cdot 2^{-\kappa}$. Set

$$\delta_N = \Gamma_k\left[(1 + \Lambda_N) C_J L_k N^{-\kappa} + 2\Lambda_N c_q \varepsilon_L\right].$$

By (5.1), $\delta_N = O\left(N^{-\kappa} \log N\right) \to 0$; let N_0 be the smallest integer with $\delta_N \le 1/2$ for $N \ge N_0$.

Remark 5.2. All quantities in δ_N are explicit, except ε_L, which can be computed. We use (5.1) and neglect ε_L. Then $\delta_N \le 1/2$ gives $N_0 \approx 2.9 \times 10^5$ for P2 and $N_0 \approx 4.2 \times 10^7$ for P1 of Section 6. In the experiments, the predicted behaviour already appears at $N = 8$. The gap comes from the worst-

case Jackson and Lebesgue estimates. Closing it needs a discrete stability argument that avoids the Hölder norm.

Theorem 5.3 (Stability)**.** Under (A1)–(A2), for $N \geq N_0$ and all $v \in \mathbb{P}_N$, $\|v\| \leq 2\Gamma_k \left\| v - I_N \mathcal{K}_\alpha^M v \right\|$. In particular (5.3) is uniquely solvable.

Proof. With $r = v - I_N \mathcal{K}_\alpha^M v$ we have $v - \mathcal{K}_\alpha v = r + (I_N - I)\mathcal{K}_\alpha v + I_N(\mathcal{K}_\alpha^M - \mathcal{K}_\alpha)v$. Apply $(I - \mathcal{K}_\alpha)^{-1}$, and bound the second term by $(1 + \Lambda_N)E_N(\mathcal{K}_\alpha v) \leq (1 + \Lambda_N)C_J L_k N^{-\kappa}\|v\|$ (Lemma 2.1(iv)) and the third by $2\Lambda_N c_q \varepsilon_L \|v\|$ (Lemma 5.1). This gives $\|v\| \leq \Gamma_k \|r\| + \delta_N \|v\|$. □

Theorem 5.4 (Error)**.** Under (A1)–(A2), for $N \geq N_0$,

$$\|y - y_N\| = \|Y - Y_N\| \leq 2\Gamma_k [(1 + \Lambda_N)E_N(Y) + 2\Lambda_N c_q \varepsilon_L \|Y\|].$$

Hence the error is $O\big(\log N\,(N^{-m} + \varepsilon_L)\big)$ if $Y \in C^m$, and $O\Big(\log N\,(\rho^{-N} + \rho_k^{-L})\Big)$ if Y is analytic in $\mathcal{E}_\rho$ and $\widetilde{k}$ is analytic.

Proof. With $e = Y - Y_N$, interpolating (2.1) and subtracting (5.2) gives $e = (Y - I_N Y) + \mathcal{K}_\alpha e + (I_N - I)\mathcal{K}_\alpha e + I_N(\mathcal{K}_\alpha - \mathcal{K}_\alpha^M)Y_N$. Proceed as in Theorem 5.3, using $\|Y_N\| \leq \|Y\| + \|e\|$; for the analytic case use $E_N(Y) \leq 2M_\rho \rho^{-N}/(\rho - 1)$ [25, Thm. 8.2]. □

Remark 5.5 (Choice of nodes). The proofs require $(1 + \Lambda_N)N^{-\kappa} \to 0$. For Jacobi–Gauss nodes with $\gamma = \max(a, b) > -1/2$, $\Lambda_N \asymp N^{\gamma+1/2}$ [10, Lemma 3.4], so the argument needs $\alpha > \gamma + 1/2$; with $\Lambda_N = O(\sqrt{N})$ this is $\mu < 1/2$, the restriction in [9]. Chebyshev nodes remove it; [16] also restricts to parameters $\leq -1/2$.

Theorem 5.6 (Conditioning)**.** Under (A1)–(A2), for $N \geq N_0$,

$$\left\| B_N^{-1} \right\|_\infty \leq 2\Gamma_k \Lambda_N, \qquad \|B_N\|_\infty \leq 1 + c_q \|k\| \Lambda_N, \qquad \kappa_\infty(B_N) \leq 2\Gamma_k \Lambda_N \big(1 + c_q \|k\| \Lambda_N\big).$$

Proof. For each s, $\sum_j \left| Q_{sj} \right| \leq \frac{q}{\Gamma(\alpha)} \sum_k w_k \left| G_{t_s}(v_k) \right| \sum_j \left| \ell_j(t_s v_k) \right| \leq c_q \|k\| \Lambda_N$. For the inverse, let $\mathbf{b}$ be given, $r = I_N \mathbf{b}$ (so $\|r\| \leq \Lambda_N \max|b_s|$), and let $v \in \mathbb{P}_N$ have nodal values $B_N^{-1}\mathbf{b}$. The polynomials $v - I_N \mathcal{K}_\alpha^M v$ and r agree at the nodes, so they are equal, and Theorem 5.3 gives $\max_s |v(t_s)| \leq \|v\| \leq 2\Gamma_k \Lambda_N \max_s |b_s|$. □

Corollary 5.7. If $k \equiv -\lambda^2$, then for $N \geq N_0(\lambda)$, $\kappa_\infty(B_N) \leq 4\Lambda_N \left(1 + c_q \lambda^2 \Lambda_N\right) = O\left(\lambda^2 \log^2 N\right)$.

Proof. Combine Theorem 5.6 with Theorem 3.1 for $\sigma = \lambda^2$. □

Remark 5.8. Here $N_0(\lambda)$ grows like $\left(\lambda^2 \log \lambda\right)^{1/\alpha}$ because $L_k \propto \lambda^2$, so for large λ the corollary does not formally cover the values of N used in Section 6.2. In the experiments the scheme was stable for all N tested; a λ-uniform discrete stability bound remains open. The unscaled matrix $\mathrm{diag}(c)B_N$ has condition number at most $\max c / \min c$ times $\kappa_\infty(B_N)$.

6. Numerical experiments

All computations are in double precision (Python, NumPy, SciPy) on one core. Times are medians of seven runs. Errors are maximum errors on 1001 equispaced points in x. We also checked P1–P4 with $N = 32$ and 64 on two finer grids: 4001 equispaced points in t, and 300 logarithmically spaced points in $\left[10^{-14}, 1\right]$. The error changed by less than a factor of two. So the grid does not miss larger errors near $x = 0$. We compare four methods with $N + 1$ unknowns:

- *HLAX*: scheme (5.2) with $M = N + 1$ [16];
- *HLAX+*: the same with $M = N + 40$;
- *JFP*: the method of [19] with Legendre polynomials in t. The integration matrix is computed exactly in $30 + N$ digits. The multiplication matrices and data coefficients use $(N + 30)$-point Gauss–Legendre quadrature in 40 digits (mpmath). Kernels must be separable, $K = \sum_r a_r(x) b_r(s)$;
- *Direct*: Chebyshev collocation in x.

Table 1: Test problems ($c \equiv 1$ unless stated).

	α	**kernel** $K(x,s)$	**data**	**reference solution**
P1	$1/2$	-9	$g = 1$	$\operatorname{erfcx}\left(3\sqrt{x}\right)$ (exact)
P2	$1/2$	$1/2\cos x$, $c = 1 + x$	$y = x^{1/2} + x$	exact
P3	$4/5$	e^{x-s}	$g = 1$	power series, 400 terms, 50 digits
P4	$2/3$	e^{-xs} (non-separable)	$g = 1 + x$	power series, 400 terms, 50 digits
P5	$3/4$	$-T^{\alpha}\lambda(Ts)$	see §6.3	HLAX+, $N = 160$

P1 is the surface temperature of a half-space cooled by Newton's law, $y = \operatorname{erfcx}\left(\lambda^2\sqrt{x}\right)$ with $\lambda = 3$. P3 is Example 4.1 of [16]. The power-series references use the recursion of Theorem 4.2. They agree to all printed digits between 300 and 400 terms. For P5 the reference was computed with the HLAX scheme itself ($N = 160$). To avoid circularity, we checked it in two independent ways. First, high-precision JFP, a different discretization, agrees with it to 2.3×10^{-13} at $N = 64$. Second, we computed the residual of the HLAX solution ($N = 64$) in the original equation. We used adaptive tanh-sinh quadrature in 30 digits and recomputed g the same way. At 25 points in [0.02,1] the residual is at most 8.4×10^{-14}. With the measured $\left\|B_{64}^{-1}\right\|_{\infty} \approx 1.85$, the error is about 1.6×10^{-13}. The rigorous bound $\Gamma_k \leq 6.1 \times 10^{4}$ only guarantees 5.1×10^{-9}.

6.1. Accuracy and cost

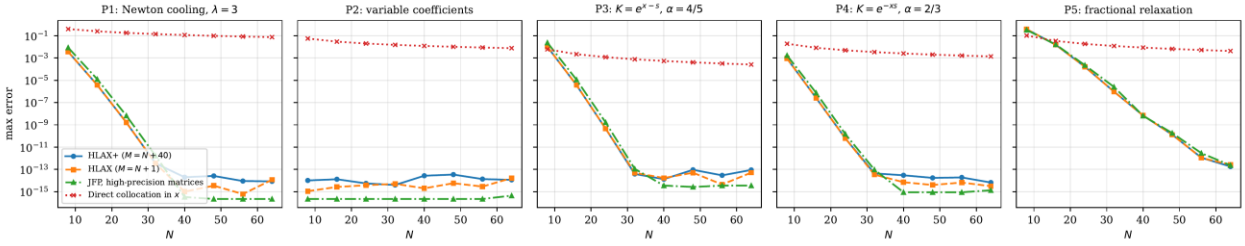


Fig. (1): Maximum error against N.

Table 2: Maximum errors at $N = 16,32,64$.

	HLAX ($M = N + 1$)			**JFP, high-precision matrices**		
	16	32	64	16	32	64
P1	3.7e-06	3.5e-13	1.2e-14	1.3e-05	1.7e-12	2.2e-16
P2	2.7e-15	5.1e-15	1.7e-14	2.2e-16	2.2e-16	4.4e-16
P3	3.7e-06	5.0e-14	5.2e-14	1.1e-05	1.1e-13	3.6e-15
P4	2.6e-07	3.4e-14	3.1e-15	7.4e-07	9.5e-14	1.3e-15
P5	1.5e-02	9.6e-07	2.6e-13	1.6e-02	2.5e-06	2.3e-13

Figure (1) and Table 2 show the following.

- HLAX and HLAX+ agree to rounding level on every problem; $M = N + 1$ quadrature points suffice, as Lemma 5.1 predicts ($L = N + 1$).
- JFP converges at a similar rate. Once converged, it reaches 10^{-15}–10^{-16} on P1–P4. This is up to two orders of magnitude below the collocation level of 10^{-14}, which is set by rounding in double precision. JFP needs high precision for the integration matrix and also for the multiplication matrices and data. With the latter in double precision, its error stopped at 10^{-12}–10^{-10} on P2–P5 and grew with N. For the non-separable kernel P4 we used a 22-term expansion of e^{-xs}.
- The direct method converges algebraically; at $N = 64$ its errors are 2.6×10^{-4} to 7.5×10^{-2}.

Table 3: Time per solve in milliseconds. Collocation: median of 7 runs; JFP: one run, mpmath.

	HLAX	**HLAX+**	**JFP: integration matrix**	**JFP: projections and solve**	**Direct**
P3, $N = 32$	0.4	0.8	42	105	0.9
P3, $N = 64$	1.6	2.6	214	554	2.5
P5, $N = 32$	0.4	0.9	39	347	0.9
P5, $N = 64$	1.8	2.8	204	917	2.6

In our implementation, the high-precision parts of JFP cost several hundred times a full HLAX solve (Table 3). For P4, with 22 separable terms, they took 11 s at $N = 64$. The integration matrix depends only on (N, p, q) and can be reused. The multiplication matrices, however, depend on the coefficients. These times come from unoptimized mpmath code, and a compiled library would be much faster. The firm conclusion is that collocation needs no extended precision.

6.2. Conditioning and stability constants

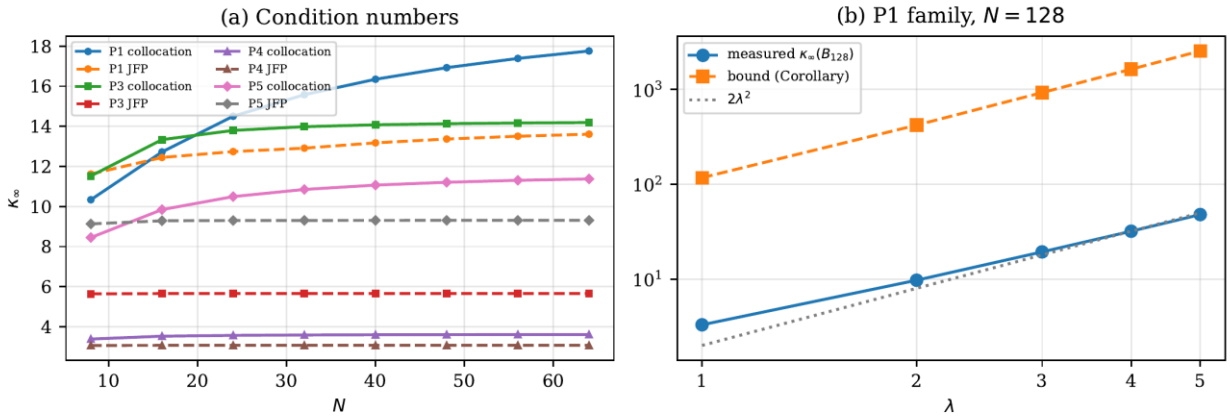


Fig. (2): (a) κ_∞ of the collocation matrix B_N and of the JFP matrix. (b) For $k \equiv -\lambda^2$, $\alpha = 1/2$, $N = 128$: measured $\kappa_\infty(B_N)$ and the bound of Corollary 5.7.

All matrices are well conditioned (Figure (2a)). $\kappa_\infty(B_N)$ grows slowly with N and stays below 18 for $N \leq 64$. The JFP condition numbers do not depend on N and are up to 2.5 times smaller. Only the collocation bound is proved.

For $k \equiv -\lambda^2$, $\lambda = 1, \ldots, 5$, the measured $\kappa_\infty(B_{128})$ is 3.3, 9.7, 19.4, 32.0, 47.9, close to $2\lambda^2$. Corollary 5.7 gives 117, 420, 924, 1630, 2538: the same λ^2 growth, with a constant 35 to 53 times too large. With $\Gamma_k = E_{1/2}\left(\lambda^2\right)$ instead of 2, the bound is 5.9×10^2, 7.5×10^9, 2.8×10^{38}, 4.9×10^{114} and 1.4×10^{275} (Figure (2b)).

Table 4 compares the two bounds for Γ_k with $\left\|B_{64}^{-1}\right\|_\infty$, a discrete proxy for Γ_k. Theorem 3.1 is nearly sharp for P1. It gains three orders of magnitude for P5. For the positive kernels P2–P4 it equals the classical bound, since $\sigma = 0$ is optimal.

Table 4: Bounds for Γ_k and the measured $\left\|B_{64}^{-1}\right\|_\infty$.

	α	$\|k\|_\infty$	$E_\alpha(\|k\|)$	**Theorem 3.1**	$\|B_{64}^{-1}\|_\infty$
P1	1/2	9.00	3.0×10^{35}	2.0	1.58
P2	1/2	0.50	1.95	1.95	1.32
P3	4/5	2.72	40.9	40.9	5.19
P4	2/3	1.00	3.87	3.87	2.22
P5	3/4	8.44	3.8×10^7	6.1×10^4	1.85

6.3. Application: fractional relaxation

Fractional relaxation models of viscoelastic and dielectric materials [1,20] lead to the Caputo problem

$$^C D^\alpha u(\tau) + \lambda(\tau)u(\tau) = \sin\tau, \qquad u(0) = 1, \qquad \tau \in [0, T],$$

with a time-dependent rate, here $\lambda(\tau) = 1 + 1/2\cos(0.4\tau)$, $T = 10$ and $\alpha = 3/4$. With $x = \tau/T$ and $y(x) = u(Tx)$ it is equivalent to (1.1) with $c \equiv 1$, $K(x,s) = -T^\alpha \lambda(Ts)$ and $g(x) = 1 +$

$T^{\alpha}\big(I^{\alpha}\sin(T\,\cdot)\big)(x)$; g was computed by a 120-point Gauss–Jacobi rule. Here $\|k\| = 8.44$ and the classical constant is $E_{3/4}(8.44) = 3.8 \times 10^{7}$. At $N = 64$ HLAX reaches 2.6×10^{-13} in 1.8 ms. Figure (3) shows the solutions for three orders: smaller α gives faster initial decay and a phase lead of the forced oscillation.

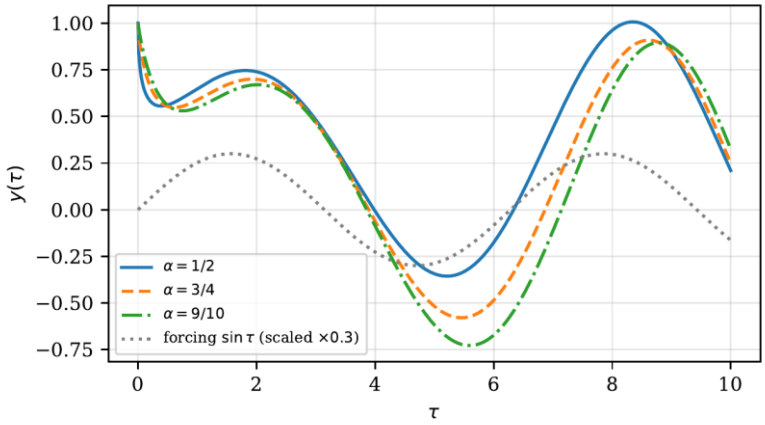


Fig. (3): Fractional relaxation: $u(\tau)$ for $\alpha = 1/2,3/4,9/10$ ($N = 64$).

7. Discussion

The scheme analysed here is not new. It is the Chebyshev case of [16], and its approximation space is that of [15] and of the JFP basis [19]. Our contributions are threefold. First, a bound on the condition number. Second, a resolvent bound that keeps it useful for large coefficients. Third, an error analysis that needs only Hölder continuity of the kernel in x and the approximability of G_t, instead of the derivative norm K^{*} of [16]. Compared with JFP, collocation accepts non-separable kernels and needs no extended precision. JFP has slightly smaller condition numbers and, with high-precision matrices, one to two more digits of accuracy.

Limitations

- The analysis requires $\alpha = p/q$; irrational orders need two-parameter Müntz spaces.
- The test set is small (five problems); in particular, oscillatory kernels and long intervals were not studied.
- Theorem 3.1 helps only when k is close to a non-positive constant: for P5 the bound remains 6.1×10^{4} against a measured 1.85. For kernels bounded below by a positive constant no gain is possible (Proposition 3.3).
- The theory is asymptotic: the explicit N_0 is of order 10^{5}–10^{7} for our problems (Remark 5.2) and grows with $\|k\|$.
- The timing comparison with JFP reflects our implementations; the accuracy comparison uses high-precision JFP matrices.
- Nonlinear equations and systems are not treated.

8. Conclusions

For weakly singular Volterra equations of rational order, transformed Chebyshev collocation has condition number $O(\log^2 N)$ with an explicit constant. For dissipative problems this constant grows like λ^2, not $\exp(\lambda^4)$, in agreement with experiment. For positive kernels the exponential growth is genuine. The method converges exponentially for analytic data with $M = N + 1$ quadrature points. It runs in double precision at millisecond cost. The results are asymptotic, and their constants are pessimistic, although their dependence on λ is correct.

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