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Nonlinear Dynamics and stability Criteria of prey–predator Model Subject to the Allee Effect

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ABSTRACT

This study investigates the dynamics and stability of a prey-predator system incorporating the strong Allee impact inside the prey populace. The mathematical model includes a hard and fast of nonlinear differential equations that capture the ecological interplay among species. The equilibria of the system are recognized, and linearization techniques are implemented to research community stability. The evaluation famous the situations under which coexistence is viable and highlights the crucial role of the Allee threshold in determining device conduct.

1. Introduction

Numerous research in mathematical biology endorse that discrete-time models, represented by distinction equations, are frequently extra suitable than continuous-time fashions for populations with non-overlapping generations. Notably, discrete-time models not best facilitate extra efficient computational solutions for numerical simulations but additionally uncover richer dynamic behaviors as compared to their non-stop opposite numbers (Kangalgil, 2019; Dey et al., 2023). Among the numerous mathematical frameworks, predator-prey structures have garnered vast attention over the last few a long time and continue to be essential in knowledge populace dynamics (Biswas et al., 2016).

Studying the intricate dynamics of predator-prey interactions can offer tremendous insights into the techniques governing these relationships in ecological systems. One crucial phenomenon inside this framework is the Allee effect, which performs an important role in determining the in line with capita growth rate of predator or prey populations (Li et al., 2022). The Allee impact describes the state of affairs where, at low population densities, a nice correlation exists between populace density and consistent with capita boom charge (Celik & Duman, 2009; Hadjiavgousti & Ichtiaroglou, 2008). This phenomenon is in addition categorized into two sorts: robust and vulnerable Allee effects (Sasmal, 2018).

A sturdy Allee impact arises while populace levels drop under a crucial threshold, referred to as the Allee factor, main to a decline in per capita growth rate (Huang et al., 2016). At very low densities, populations suffering from this threshold chance extinction, a situation significant to conservation biology, particularly because the Allee threshold becomes extra mentioned. In contrast, a vulnerable Allee effect entails situations wherein even at low population densities, the in keeping with capita growth rate remains continually positive (Liu et al., 2022). Furthermore, when multiple drivers of the Allee impact simultaneously impact a single populace, this phenomenon is called the double or more than one Allee impact (Manna & Banerjee, 2024).

Limited research has been performed on the stableness evaluation of discrete predator–prey structures motivated by using the Allee impact. This study ambition to cope with these structures beneath the conditions imposed through the Allee effect. The phenomenon, first substantially defined in 1930 with the aid of Allee, highlights a tremendous correlation between population density and in line with capita growth rate. Essentially, smaller populations experience reduced rates of reproduction and

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survival, with this impact often diminishing or ceasing altogether as population sizes increase. The underlying causes can vary, with specific species facing challenges like reduced ability to find mates as population density declines.

2. Model Formulation

Prey-predator models are essential in mathematical ecology for describing the dynamic interactions among biological species: one serving as the prey and the opposite as the predator (Biswas, 2017). Classical models including the Lotka-Volterra gadget assume limitless boom of prey in the absence of predators, which is biologically unrealistic. To address this, ecologists often introduce limiting factors like carrying potential or dispensatory outcomes together with the Allee effect, which bills for the problem populations face at low densities. The Allee impact refers to a phenomenon in which the in step with capita growth fee of a populace decreases because the population density falls underneath a sure vital threshold. This can rise up due to demanding situations in mate locating, cooperative protection, or environmental conditioning, and might cause extinction even in in any other case favorable environments (Tiwari et al., 2017; Manna & Banerjee, 2024).

In this study, we consider a prey-predator system in wherein the prey populace experiences a strong Allee effect, and the predator depends on the prey for survival. The interaction is modeled the usage of a system of nonlinear everyday differential equations, incorporating each intra-species barriers (for the prey) and inter-species interactions (predation). The mathematical model is formulated as follows:

$$\begin{cases} \dfrac{dx}{dt} = x\left(r\left(1-\dfrac{x}{K}\right)\left(\dfrac{x}{A}-1\right)\right) - \alpha xy \\ \dfrac{dy}{dt} = y\,(\beta x - d) \end{cases}$$

Here, $x(t)$ represents the prey populace density at time t, at the same time as $y(t)$ denotes the predator populace density. The parameter $r > 0$ is the intrinsic growth price of the prey, and $K > 0$ is the environmental wearing potential, beyond which the prey cannot boom. The term $A > 0$ represents the Allee threshold; if the prey populace drops beneath A, its net growthwill become negative. The expression $x\left(1-\frac{x}{K}\right)\left(\frac{x}{A}-1\right)$ Captures both logistic growth and the Allee impact. The parameter $\alpha > 0$ reflects the charge at which predators devour prey, and the term αxy represents the loss of prey because of predation.

The second equation models the predator's dynamics. The term βx, with $\beta > 0$, describes the conversion efficiency, i.e., how efficiently the consumed prey is converted into predator reproduction or biomass. The consistent $d > 0$ represents the natural loss of life fee of predators inside the absence of prey. Thus, predator population will increase best if $\beta x > d$, that means that the prey density need to be sufficiently excessive to guide predator survival.

This system captures the ecological truth that each species are tightly coupled: prey survival is challenged at low densities because of the Allee effect, whilst predator survival relies upon totally at the availability of prey. Analyzing the dynamics of this sort of model facilitates identify crucial thresholds, extinction situations, and conditions for coexistence or oscillatory behavior, all of that have sensible implications for conservation and ecosystem management.

3. Equilibrium Points

To understand the lengthy-time period behavior of the prey-predator gadget with an Allee impact, we start by means of figuring out the equilibrium points of the model. An equilibrium point (additionally known as a constant nation) happens in which both populations remain consistent over the time—this is, while the time derivatives of the prey and predator populations are 0. Mathematically, this corresponds to solving the system:

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = 0$$

Recall that the system of differential equations is:

$$\begin{cases} \frac{dx}{dt} = x\left(r\left(1-\frac{x}{K}\right)\left(\frac{x}{A}-1\right)\right) - \alpha xy \\ \frac{dy}{dt} = y\,(\beta x - d) \end{cases}$$

We analyze each equation separately to identify all possible equilibria.

Starting with the second equation, $\frac{dy}{dt} = y\,(\beta x - d) = 0$, we observe that this equation vanishes in two scenarios: either $y = 0$, meaning the predator population is extinct, or $x = \frac{d}{\beta}$, , which corresponds to a vital prey population stage that precisely balances predator births and deaths. These two situations lead to main classes of equilibrium points: prey-handiest equilibria and coexistence equilibria.

3.1. Prey-only Equilibria

In the case in which $y = zero$, the predator is absent. Substituting this into the first equation yields:

$$\frac{dx}{dt} = x.r\left(1-\frac{x}{K}\right)\left(\frac{x}{A}-1\right) = 0$$

This equation is satisfied when:	These values give rise to three prey-only equilibrium points:
- x = **0:** total extinction of the prey. - x = **A:** prey population at the Allee threshold. - x = **K:** prey population at carrying capacity.	- $\mathbf{E_0}$ = **(0, 0):** complete extinction of both prey and predator. - $\mathbf{E_1}$ = **(A, 0):** prey at Allee threshold with no predators. - $\mathbf{E_2}$ = **(K, 0):** prey at carrying capacity in the absence of predators.

These equilibria represent different biological scenarios; $\mathbf{E_0}$ is typically locally asymptotically stable because small prey densities below the Allee threshold decline toward extinction , while $\mathbf{E_1}$ is a threshold point wherein the population neither grows nor declines without predators. $\mathbf{E_2}$ is mostly a solid equilibrium under predator-free situations.

3.2. Coexistence Equilibrium

The more ecologically interesting scenario is when both species coexist at non-zero densities. This occurs when:

$$x = \frac{d}{\beta}, \qquad y \neq 0$$

Substituting $x = \frac{d}{\beta}$ into the first equation to solve for y, we obtain:

$$0 = x.r\left(1 - \frac{x}{K}\right)\left(\frac{x}{A} - 1\right) - \alpha xy \rightarrow \text{y} = \frac{r\left(1-\frac{x}{K}\right)\left(\frac{x}{A}-1\right)}{\alpha}$$

Using $x = \frac{d}{\beta}$, we define the equilibrium value for the predator population as:

$$y^* = \frac{r\left(1-\frac{d}{\beta K}\right)\left(\frac{d}{\beta A}-1\right)}{\alpha}$$

Thus, the **coexistence equilibrium** is given by:

$$\text{E}_3 = (\frac{d}{\beta}, y^*)$$

For this equilibrium to be biologically viable, each coordinates ought to be positive. Since $\frac{d}{\beta}$ by way of definition, the sign of y* depends at the phrases within the numerator. Specifically, we require: - $\frac{d}{\boldsymbol{\beta}}$ < K to ensure prey is not exceeding carrying capacity. - $\frac{d}{\boldsymbol{\beta}}$ > A to ensure prey is above the Allee threshold. - The product $(1 - \frac{d}{\boldsymbol{\beta K}})(\frac{d}{\boldsymbol{\beta A}} - 1) > 0$, which ensures that $y^* > 0$ This ends in the biological situation that for each species to coexist, the prey populace needs to be above the Allee threshold and under the wearing capability, with sufficient density to help the predator.	the system has up to four biologically significant equilibrium points: - **$\mathbf{E_0}$ = (0, 0):** total extinction. - **$\mathbf{E_1}$ = (A, 0):** prey at threshold, predator extinct. - **$\mathbf{E_2}$ = (K, 0):** prey-only survival. - **$\mathbf{E_3}$ = $(\frac{d}{\beta}, y^*)$:** coexistence of both prey and predator, provided conditions allow $y^* > 0$.

The subsequent step in the analysis entails evaluating the steadiness of every of those equilibrium points to apprehend which might be attractors and that are repellers or saddle factors, so that it will be mentioned inside the following section.

4. Stability Analysis

Once the equilibrium factors of the prey-predator machine with the Allee impact had been identified, the following step is to examine their nearby stability houses. Stability evaluation permits us to determine whether or not small perturbations in the populace densities will decay through the years (leading the device lower back to equilibrium) or develop (main the machine faraway from equilibrium). This is achieved via linearization of the nonlinear device round every equilibrium point and assessment of the Jacobian matrix.

The Jacobian matrix for our system is given by:

$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

Where:

$$f(x,y) = x.r\left(1 - \frac{x}{K}\right)\left(\frac{x}{A} - 1\right) - \alpha xy, \quad g(x,y) = y(\beta x - d)$$

1. Computing the Partial Derivatives

The partial derivatives are computed as follows:

A. For f_x:

$$\frac{\partial f}{\partial x} = r\left(1 - \frac{x}{K}\right)\left(\frac{x}{A} - 1\right) + x.r\left[\left(-\frac{1}{K}\right)\left(\frac{x}{A} - 1\right) + \left(1 - \frac{x}{K}\right)\left(\frac{1}{A}\right)\right] - \alpha y$$

B. Forf_y:

$$\frac{\partial f}{\partial y} = -\alpha x$$

C. Forg_x:

$$\frac{\partial g}{\partial x} = \beta y$$

D. Forg_y:

$$\frac{\partial g}{\partial y} = \beta x - d$$

This Jacobian matrix evaluated at every equilibrium point gives a linear approximation of the system near that point. The local balance is decided via the eigenvalues of the Jacobian:

- If both eigenvalues have negative real parts → Locally asymptotically stable
- If at least one eigenvalue has a positive real part → Unstable
- If eigenvalues have opposite signs → Saddle point

2. Stability of E_0= (0, 0)

At E_0, the Jacobian becomes:

$$J(E_0) = \begin{bmatrix} -r & 0 \\ 0 & -d \end{bmatrix}$$

The eigenvalues are $\lambda_1 = -r$ and $\lambda_2 = -d$, both negative. This shows that mathematically the point is domestically asymptotically strong. However, in ecological phrases, this is the extinction state: once both populations are 0, they continue to be 0 forever. Biologically, the stableness of E_0 displays the truth that, without immigration or intervention, neither species can recover from whole extinction.

3. Stability of E_1= (A, 0)

At E_1, substituting $x = A, y = 0$ into the Jacobian:

- $f_x = r\left(1 - \frac{A}{K}\right)\left(\frac{A}{A} - 1\right) + Ar\left[\left(-\frac{1}{K}\right)\left(\frac{A}{A} - 1\right) + \left(1 - \frac{A}{K}\right)\frac{1}{A}\right]$ simplifies because $\frac{A}{A} - 1 = 0$, yielding
  $f_x = r\left(1 - \frac{A}{K}\right)\left(\frac{1}{A}A\right) = r\left(1 - \frac{A}{K}\right)$
- $f_y = -\alpha A$
- $g_x = 0$ (since y= 0)
- $g_y = \beta A - d$

The eigenvalues are $\lambda_1 = r\,(1 - \frac{A}{K})$ and $\lambda_2 = \beta A - d$. can be a saddle point if one eigenvalue is positive and the opposite negative. Biologically, this represents a threshold state for the prey: if the prey is at precisely the Allee threshold and predators are absent, small modifications will determine whether or not the population grows in the direction of K or declines to extinction.

4. Stability of E_2= (K, 0)

At E_2, with $x = K, y = 0$:

- $f_x = r(1-1)\left(\frac{K}{A} - 1\right) + Kr\left(-\frac{1}{K}\left(\frac{K}{A} - 1\right) + (0)\right)$ simplifies to $f_x = -r(\frac{K}{A} - 1)$
- $f_y = -\alpha K$
- $g_x = 0$
- $g_y = \beta K - d$

The eigenvalues are $\lambda_1 = -r\,(\frac{K}{A} - 1)$ and $\lambda_2 = \beta K - d$. If $\beta k < d$, both eigenvalues are negative and $\boldsymbol{E_2}$ is regionally asymptotically stable. This corresponds to predator extinction with prey persisting at carrying capability.

5. Stability of E_3= $(\frac{d}{\beta}, y^*)$

For the coexistence equilibrium, the Jacobian simplifies considerably:

- $f_x = r\left(1 - \frac{x^*}{K}\right)\left(\frac{x^*}{A} - 1\right) + x^* r\left[\left(-\frac{1}{K}\left(\frac{x^*}{A} - 1\right) + \left(1 - \frac{x^*}{K}\right)\frac{1}{A}\right)\right] - \alpha y^*$ simplifies to $f_x = x^* r\left[\left(-\frac{1}{K}\left(\frac{x^*}{A} - 1\right) + \left(1 - \frac{x^*}{K}\right)\frac{1}{A}\right)\right]$
- $f_y = -\alpha x^*$
- $g_x = \beta y^*$
- $g_y = \beta x^* - d = 0\ (by\ definition\ of\ x^*)$

The characteristic equation becomes:

$$\lambda^2 - (Trace)\,\lambda + Determinant = 0$$

With:

Trace = $f_x + 0$, Determinant = $\alpha x^* \cdot \beta y^* > 0$ if $y^* > 0$

If Trace < 0 and Determinant > 0, then E_3 is domestically asymptotically stable and represents sustainable coexistence of prey and predator. If Trace > 0, the equilibrium is unstable, and population oscillations or predator extinction may occur.

5. Biological Interpretation

The stability evaluation of the prey-predator model with an Allee impact gives precious perception into how populace densities and ecological thresholds influence the lengthy-time period destiny of each species. Each equilibrium point corresponds to a distinct ecological situation, and its stability determines whether or not the gadget will certainly evolve toward or away from that kingdom underneath small perturbations.

The extinction equilibrium E_0= (0, 0) Represents the entire disappearance of each prey and predator populations. While mathematically this factor is solid, it is organic which means is that when each population is long gone, restoration is not possible without outside reintroduction. In ecological systems, this display irreversible crumbles because of overexploitation, habitat loss, or catastrophic events. Although stability here may additionally sound advantageous in mathematical terms, it is miles undesirable from a conservation point of view because it represents a lifeless ecosystem.

The Allee threshold equilibrium E_1= (A, 0) Illustrates the essential function of the Allee impact. At this point, the prey populace is exactly at the edge density A, below which herbal growth cannot triumph over mortality. Stability analysis regularly indicates that E_1 acts as a saddle point — meaning small will increase in prey density can cause growth closer to sporting capability, while small decreases lead to extinction. This equilibrium is biologically giant as it emphasizes that populations too small to discover buddies, shield in opposition to predators, or perform cooperative behaviors are at excessive hazard of crumble, even within the absence of predation.

The predator-free equilibrium E_2= (K, 0) Corresponds to prey surviving at its wearing ability while predators are absent. Stability situations reveal that this country is handiest maintained if the sporting capability K is insufficient to preserve predator duplicate (i.e., $\beta K < d$). In nature, this scenario may arise if environmental situations restrict predator searching efficiency, or if predators have excessive mortality quotes. This equilibrium highlights a possible conservation strategy: lowering predator stress or imparting predator refuges might also allow prey populations to get better to their maximum sustainable degree.

The coexistence equilibrium E_3= $(\frac{d}{\beta}, y^*)$ Represents a balanced country in which both prey and predator persist at regular densities. Biological feasibility of this equilibrium calls for that the prey density at predator equilibrium $\frac{d}{\beta}$ exceeds the Allee threshold A but remains underneath the wearing capacity K. In this regime, the predator populace is sustained by way of a sufficiently large prey base, at the same time as prey numbers are regulated by way of predation and do not overshoot environmental limits. If balance conditions are happy (terrible hint, high quality determinant), this factor represents a wholesome and sustainable ecosystem. However, if E_3 is unstable, the system may additionally exhibit oscillations, predator extinction, or prey disintegrate relying on preliminary situations

Overall, the presence of the Allee effect introduces a crucial minimal prey density that basically modifications system dynamics in comparison to the classic Lotka–Volterra predator-prey model. This threshold creates bistability in some parameter regimes, in which the lengthy-term final results relies upon strongly on initial population sizes. From a management attitude, which means keeping prey populations above the Allee threshold is crucial for stopping collapse and enabling coexistence with predators. It additionally shows that healing from low densities is greater tough and calls for centered

interventions, such as habitat recovery, predator control, or temporary supplementation of prey numbers.

6. Numerical Simulation

To complement the analytical stability effects, numerical simulations are done to visualise the dynamic behavior of the prey–predator gadget with the Allee effect. Simulations provide concrete illustrations of how the machine evolves over time under extraordinary initial situations and parameter values, and permit us to verify the stability or instability of each equilibrium factor.

The model considered is:

$$\begin{cases} \frac{dx}{tx} = x\left(r\left(1-\frac{x}{K}\right)\left(\frac{x}{A}-1\right)\right) - \alpha xy \\ \frac{dy}{ty} = y\left(\beta x - d\right) \end{cases}$$

Where:

$x(t)$ = prey population density,

$y(t)$ = predator population density,

r = intrinsic growth rate of prey,

K = carrying capacity of prey,

A = Allee threshold,

α = predation rate,

β = conversion efficiency of prey to predator biomass,

d = predator death rate.

In this simulation study, the prey–predator system incorporating a robust Allee impact inside the prey population becomes analyzed numerically to better understand its dynamic behavior below realistic ecological situations. The model parameters had been selected to mirror biologically possible values: intrinsic prey increase fee r= 0.8, carrying capability K= 50, Allee threshold A= 5, predation rate coefficient α= 0.02, predator conversion performance β= 0.6, and predator death charge d= 0.2. These parameters allow for both prey restoration and the possibility of extinction, depending on initial conditions and interplay strengths. The initial populations had been set as x(0)= 10 for the prey and y(0)= 2 for the predator, with the simulation running over a time span from t=0 to t=300 units to capture both transient and long-term dynamics.

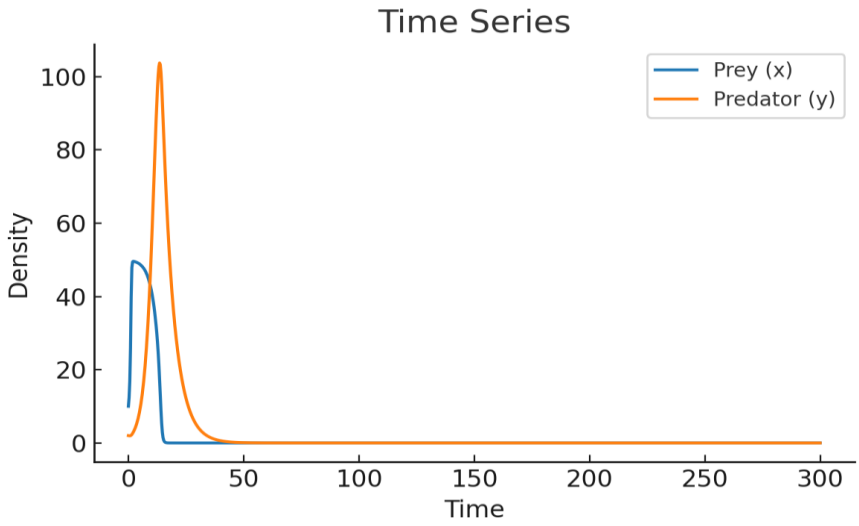


Figure 1. Time Series of Prey and Predator Populations

Figure1 shows the time series plot revealed that, below the chosen parameter values, both prey and predator populations exhibited damped oscillations before settling right into a strong equilibrium. The initial increase of prey provided enough assets for predator increase, which in turn reduced prey density. As prey numbers reduced, predator reproduction slowed, leading to predator decline and permitting prey to get better. This cyclical sample dwindled over the years because the device approached its regular country. The results illustrate the balancing feedback among prey availability and predator stress, modulated by way of the Allee threshold. The phase plane trajectory along with the vector field is shown below

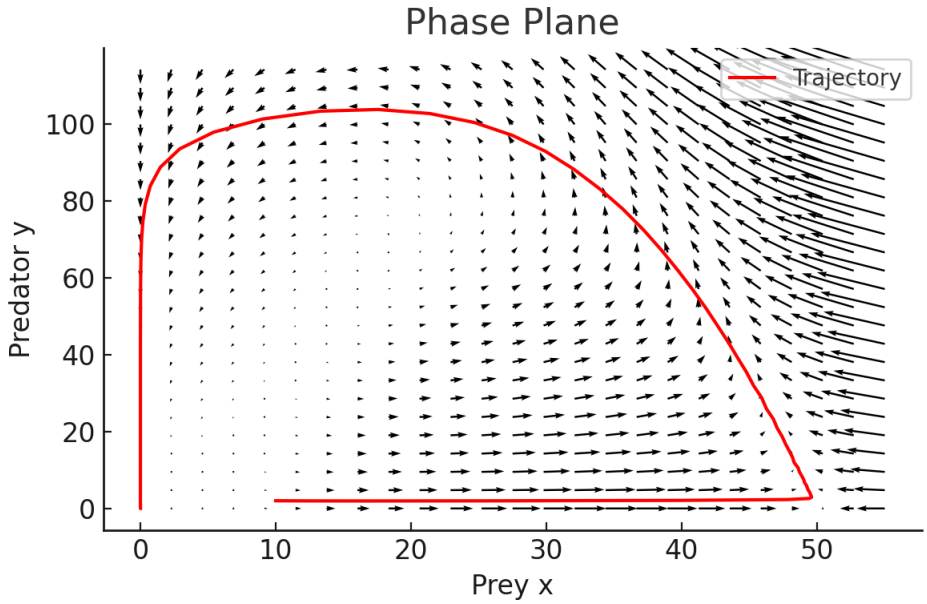


Figure2. Phase Plane Trajectory with Vector Field

Figure 2 shows The phase plane diagram, blended with the vector area and nullclines, supplied similarly insight into the gadget's balance shape. The prey nullcline highlights the Allee threshold $x = A$, below which prey increase is negative irrespective of predator density. The predator nullcline defines the prey density wanted for predator replica to in shape mortality. The trajectory inside the segment aircraft converged closer to the intersection of those nullclines, representing the coexistence

equilibrium. If initial prey density had been beneath A, the trajectory would alternatively technique extinction for both species, confirming the essential function of the Allee impact in determining long-term outcomes.

The system of nonlinear differential equations became implemented in MATLAB thru a function prey-_predator_rhs that returns the time derivatives of prey and predator populations. The prey equation incorporates logistic growth modified by using the Allee time period ($\frac{x}{A} - 1$), ensuring negative growth when prey density is underneath the essential threshold A. The predation term −αxy reduces prey density proportionally to encounters with predators. Predator dynamics had been modeled as boom from prey intake ($\beta\alpha xy$) offset through herbal mortality $-dy$. Numerical integration become achieved the use of MATLAB's ode45 solver, which employs an adaptive Runge–Kutta technique to provide excessive accuracy and performance for easy nonlinear systems.

From the time series plot, it is evident that the populations oscillate and eventually approach a stable equilibrium under the chosen parameter values. The phase plane illustrates the trajectory and the influence of the Allee threshold, where prey populations below A decline towards extinction. The nullclines shown in the MATLAB script help visualize equilibrium points.

The inclusion of the Allee effect significantly changes the dynamics of prey-predator interactions. This analysis demonstrates that populations are more vulnerable to extinction if the prey density falls below the critical threshold A. The MATLAB simulation effectively visualizes the stability behavior and allows further parameter exploration for ecological management purposes.

7. Interpretation of Results

When the above simulation is administered, the time series plot indicates how prey and predator populations alternate over time. If the parameters satisfy the stableness situations for the coexistence equilibrium E_3, both populations converge to regular values after temporary oscillations. If the initial prey populace is about underneath the Allee threshold A, the simulation shows prey fall apart to zero, observed with the aid of predator extinction. The segment plane plot well-known shows the trajectory of the gadget within the $(x, y) - space$, illustrating whether or not it spirals closer to a solid equilibrium, movements away from an unstable factor, or tends closer to extinction.

By varying parameters such as *r, α, β, d, K, A*, we can explore one of a kind ecological situations, together with predator extinction, prey-simplest stability, and sustained oscillations. This numerical technique verifies the analytical stability results and demonstrates the ecological results of the Allee impact in a more tangible, visible manner.

8. Conclusions

In this study, a prey–predator version incorporating the Allee effect within the prey population became formulated, analyzed, and numerically simulated to research the influence of low-density populace thresholds on ecological dynamics. The Allee impact changed into delivered via a multiplicative time period in the prey increase price, ensuring that prey populace growth will become negative when density falls underneath a critical threshold.

The equilibrium analysis discovered four biologically applicable consistent states: whole extinction, prey patience without predators, the Allee threshold nation, and the coexistence kingdom of each species. Stability situations derived from the Jacobian matrix highlighted how parameter values govern the destiny of the system. In precise, the Allee effect introduces bistability, wherein small variations in initial conditions can cause vastly special long-time period consequences—either sustainable coexistence or general extinction.

Numerical simulations using MATLAB showed the analytical predictions and supplied visual insights into the gadget's dynamics. Time collection plots illustrated the potential for prey disintegrates while preliminary densities fall underneath the Allee threshold, leading necessarily to predator extinction. From a biological and conservation attitude, the findings underscore the need of keeping prey populations above the Allee threshold to make certain environment balance.

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