The Identity Graph of the Direct Product $S_3 \times A_4$

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ABSTRACT

The identity graph of a finite group joins the identity to every other element and joins two non-identity elements whenever their product is the identity. This paper gives a complete description of the identity graph of the direct product $G = S_3 \times A_4$, a group of order 72. The graph is made up of 15 lines and 28 triangles sharing the identity vertex; it has 99 edges and independence number 43, and exactly 13 of its triangles are subgroups of G. Five vertex colourings are examined, by element order, conjugacy class, Sylow subgroups, centre and parity, and in each case the independent colour classes are determined; for instance, 10 of the 12 conjugacy classes are independent. All counts follow from general formulas for direct products, which reduce the work to the factors S_3 and A_4. It is also shown that G has no normal clique and that, among the 50 groups of order 72, only G and $D_{12} \times C_6$ have this identity graph.

1. Introduction

Vasantha Kandasamy and Smarandache [9] attached to every finite group a graph, called its identity graph, in which the identity is adjacent to all other elements and two non-identity elements are adjacent precisely when their product is the identity. Graphs built from inverse pairs in a similar spirit were considered by Chalapathi and Kiran Kumar [4] and by Alfuraidan and Zakariya [2]. Lawan [7] examined the identity graph of S_4 by means of five colourings of its vertices, and in [1] these colourings were worked out for every symmetric group S_n.

The object of the present paper is the direct product

$$G = S_3 \times A_4,$$

which has order 72. Direct products are the natural case to treat after the symmetric groups, since both the identity graph and its colourings can be read off from the two factors. The required product formulas are established in Section 4 and then applied to G. Each element of G is given a label $\beta_1, \dots, \beta_{72}$ taken from Table A.1 in Appendix A, so every colour class below can be written out element by element.

The notation follows [1]. For a finite group H with identity e we put

$$SI(H) = \{h \in H : h^2 = e\}, \qquad s(H) = |SI(H)|, \qquad m(H) = |H| - s(H),$$

and we write $c(H)$ for the number of elements $h \in H$ with $h^3 = e$. The main results are the following three theorems.

Theorem 1.1 (Structure of $\Omega(G)$): Let $G = S_3 \times A_4$.

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(i) $s(G) = 16$ and $m(G) = 56$. Consequently $\Omega(G)$ is the union of 15 lines and 28 triangles, all passing through the identity, and it has 99 edges. The identity has degree 71, each of the 15 involutions has degree 1, and each of the remaining 56 vertices has degree 2.

(ii) The clique number and the chromatic number of $\Omega(G)$ are both 3, the domination number is 1 and the independence number is 43; there are 2^{28} independent sets of size 43.

(iii) Exactly 13 of the triangles are subgroups of G, each isomorphic to C_3, and the other 15 are not. In particular G is not graphically nice.

Table 1: The five colourings of $\Omega(G)$, $G = S_3 \times A_4$.

Colouring	Colour classes	Sizes	Independent classes
Order	orders 1, 2, 3, 6	1, 15, 26, 30	orders 1 and 2
Conjugacy	the 12 classes $C_1, \ldots, C_{12}$ of Table 3	1, 3, 3, 9, 2, 4, 4, 8, 8, 12, 12, 6	all except C_5 and C_{12}
Sylow	non-identity 2-elements, non-identity 3-elements, uncoloured	15, 26, 30	the 2-elements
Centre	$\{e\}$ and $G\backslash\{e\}$	1, 71	$\{e\}$
Parity	even and odd elements	36, 36	none

Theorem 1.2 (Colourings): The colour classes of the five colourings of $\Omega(G)$, their sizes and their independence are as listed in Table 1. Moreover:

(i) exactly 6 of the 12 conjugacy classes are real, and there are 4 triangles whose two non-identity vertices lie in the same class;

(ii) G has 3 Sylow 2-subgroups, each isomorphic to C_2^3, and 4 Sylow 3-subgroups, each isomorphic to C_3^2; two distinct Sylow 2-subgroups meet in a subgroup of order 4, and two distinct Sylow 3-subgroups meet in a subgroup of order 3;

(iii) the even elements carry 16 triangles and the odd elements carry 12.

Theorem 1.3 (Normal subgroups and isomorphisms):

(i) G has exactly nine normal subgroups, of orders 1,3,4,6,12,12,24,36,72, and G has no normal clique.

(ii) If H is a group of order 72, then $\Omega(H) \cong \Omega(G)$ if and only if $H \cong G$ or $H \cong D_{12} \times C_6$.

(iii) $\Omega(G)$ is not isomorphic to the identity graph of any dihedral group.

Part (ii) of Theorem 1.3 shows that the identity graph by itself does not determine G. The colourings of Theorem 1.2 do, however, tell G apart from $D_{12} \times C_6$ (Remark 7.1).

The rest of the paper is arranged as follows. Section 2 gives the definitions and the data for the two factors. Section 3 contains three lemmas that hold in every finite group, and Section 4 proves the product formulas. Theorems 1.1, 1.2 and 1.3 are proved in Sections 5, 6 and 7, and Section 8 explains how the results were checked by computer.

2. Preliminaries

2.1. Definitions

All groups considered are finite. For an element h we denote its order by $o(h)$, and an element of order 2 is called an involution. We set $MI(H) = H \backslash SI(H)$, so that $|MI(H)| = m(H)$. The dihedral group of order $2N$ is denoted D_{2N} and the cyclic group of order N is denoted C_N. A line is a pair of adjacent vertices, a triangle is a set of three pairwise adjacent vertices, and a set of vertices is independent when no two of its members are adjacent.

Definition 2.1 ([9]). The identity graph $\Omega(H)$ has vertex set H, and two distinct vertices x, y are adjacent if and only if one of them equals e, or $x \neq e$, $y \neq e$ and $xy = e$.

Definition 2.2 ([7]). The colouring of $\Omega(H)$ by a property of elements assigns two vertices the same colour exactly when they share the same value of that property. Its colour classes form a partition of H, and the question to be settled is which of these classes are independent.

Definition 2.3. A conjugacy class K is real if $h^{-1} \in K$ for every $h \in K$. A p-element is an element whose order is a power of p (the identity included). We write $\pi_p(H)$ for the number of p-elements of H and $n_p(H)$ for the number of its Sylow p-subgroups.

Definition 2.4 ([9, 7]). A normal clique of H is a family of at least two normal subgroups of H, each different from $\{e\}$ and from H, such that any two of them intersect in $\{e\}$ and their union is H. The group H is graphically nice if every triangle of $\Omega(H)$ is a subgroup.

2.2. The factors

The direct product $H_1 \times H_2$ is the set of pairs (a, b) with componentwise multiplication; its identity is (e, e) and $(a, b)^{-1} = \left(a^{-1}, b^{-1}\right)$. In A_4 let V_4 be the Klein four-group formed by the identity and the double transpositions, let K_1 be the conjugacy class of (1 2 3), and let $K_2 = K_1^{-1}$:

$$K_1 = \{(1\,2\,3), (1\,4\,2), (1\,3\,4), (2\,4\,3)\}, \qquad K_2 = \{(1\,3\,2), (1\,2\,4), (1\,4\,3), (2\,3\,4)\}.$$

Table 2 records the conjugacy classes of the two factors together with the data needed later; all entries are standard [6].

Table 2: Conjugacy classes of the factors.

Group	Class	Size	Order	Real	$h^2 = e$	Sign
S_3	$\{e\}$	1	1	yes	yes	+
	transpositions	3	2	yes	yes	−
	3-cycles	2	3	yes	no	+
A_4	$\{e\}$	1	1	yes	yes	+
	$V_4 \backslash \{e\}$	3	2	yes	yes	+
	K_1	4	3	no	no	+
	K_2	4	3	no	no	+

From Table 2 we read off $s(S_3) = 4$, $s(A_4) = 4$, $c(S_3) = 3$ and $c(A_4) = 9$. In addition $n_2(S_3) = 3$, $n_3(S_3) = 1$, $n_2(A_4) = 1$ (the subgroup V_4) and $n_3(A_4) = 4$, and both factors have trivial centre.

3. Three Lemmas for Arbitrary Groups

The next three lemmas appear in [1]. Since their proofs are short, we give them here for completeness.

Lemma 3.1: $\Omega(H)$ is the union of $s(H) - 1$ lines and $m(H)/2$ triangles, all of which contain e. The identity has degree $|H| - 1$, every involution has degree 1, every element of $MI(H)$ has degree 2, and $\left|E(\Omega(H))\right| = |H| - 1 + m(H)/2$.

Proof: A vertex $h \neq e$ is adjacent to e, and its only other possible neighbour is h^{-1}, which is a neighbour exactly when $h^{-1} \neq h$. Hence every involution h gives a line $\{e, h\}$, while the $m(H)$ elements of $MI(H)$ split into $m(H)/2$ pairs $\{h, h^{-1}\}$, each giving a triangle. A line has one edge and a triangle has three, so the number of edges is $(s(H) - 1) + 3m(H)/2 = |H| - 1 + m(H)/2$. ▫

Lemma 3.2: $\Omega(H) \cong \Omega(H')$ if and only if $|H| = |H'|$ and $s(H) = s(H')$.

Proof: Isomorphic graphs have equal numbers of vertices and of edges, so $|H| = |H'|$, and then $s(H) = s(H')$ by Lemma 3.1. Conversely, any bijection that sends e to e, involutions to involutions and inverse pairs $\{h, h^{-1}\}$ to inverse pairs is a graph isomorphism. ▫

Lemma 3.3: A conjugacy class K contains an edge of $\Omega(H)$ if and only if K is real and $K \subseteq MI(H)$.

Proof: Conjugate elements have the same order, so either $K \subseteq MI(H)$ or $K \subseteq SI(H)$. An edge lying inside a class $K \neq \{e\}$ must have the form $\{h, h^{-1}\}$ with $h \in MI(H)$. If K is real and $K \subseteq MI(H)$, such a pair lies in K. If $K \subseteq SI(H)$, no such pair exists. If K is not real, then K^{-1} is another conjugacy class, hence disjoint from K, and again no such pair exists. ▫

4. Product Formulas

In this section H_1 and H_2 denote finite groups.

Lemma 4.1:

(a) $o(a, b) = \mathrm{lcm}\big(o(a), o(b)\big)$.
(b) $SI(H_1 \times H_2) = SI(H_1) \times SI(H_2)$; hence $s(H_1 \times H_2) = s(H_1)\, s(H_2)$.
(c) $c(H_1 \times H_2) = c(H_1)\, c(H_2)$, and $\pi_p(H_1 \times H_2) = \pi_p(H_1)\, \pi_p(H_2)$ for every prime p.

Proof: For each $d \geq 1$ we have $(a, b)^d = \left(a^d, b^d\right)$, and this equals the identity exactly when $a^d = e$ and $b^d = e$. This proves (a); taking $d = 2$ and $d = 3$ gives (b) and the first claim of (c). By (a), $o(a, b)$ is a power of p exactly when both $o(a)$ and $o(b)$ are, which gives the second claim of (c). ▫

Lemma 4.2: The conjugacy classes of $H_1 \times H_2$ are the sets $K \times L$, where K is a class of H_1 and L is a class of H_2. The class $K \times L$ is real if and only if both K and L are real, and $K \times L \subseteq SI(H_1 \times H_2)$ if and only if $K \subseteq SI(H_1)$ and $L \subseteq SI(H_2)$.

Proof: Since $(x, y)(a, b)(x, y)^{-1} = \left(xax^{-1}, yby^{-1}\right)$, the elements (a, b) and (a', b') are conjugate exactly when a, a' are conjugate in H_1 and b, b' are conjugate in H_2. As $(K \times L)^{-1} = K^{-1} \times L^{-1}$, the class is real exactly when both factors are real. The final statement follows from Lemma 4.1(b). ▫

Lemma 4.3: The Sylow p-subgroups of $H_1 \times H_2$ are exactly the subgroups $P_1 \times P_2$, where P_i is a Sylow p-subgroup of H_i. In particular $n_p(H_1 \times H_2) = n_p(H_1)\, n_p(H_2)$.

Proof: $P_1 \times P_2$ is a p-group whose order is the p-part of $|H_1|\,|H_2|$, so it is a Sylow p-subgroup. Conversely, let P be a Sylow p-subgroup of $H_1 \times H_2$ and let P_1, P_2 be its images under the two projections. Both are p-groups and $P \leq P_1 \times P_2$; because P already has the largest order a p-subgroup can have, $P = P_1 \times P_2$, and then P_1 and P_2 are Sylow p-subgroups. Distinct pairs (P_1, P_2) give distinct subgroups. ▫

Lemma 4.4: $Z(H_1 \times H_2) = Z(H_1) \times Z(H_2)$.

Proof: (a, b) commutes with every (x, y) exactly when a commutes with every $x \in H_1$ and b commutes with every $y \in H_2$. ▫

5. Proof of Theorem 1.1

(i) Lemma 4.1(b) and Table 2 give

$$s(G) = s(S_3)\, s(A_4) = 4 \cdot 4 = 16, \qquad m(G) = 72 - 16 = 56.$$

By Lemma 3.1, $\Omega(G)$ has $16 - 1 = 15$ lines and $56/2 = 28$ triangles, the degrees are as stated, and the number of edges is $71 + 28 = 99$.

(ii) A vertex other than e has at most one neighbour apart from e, so no four vertices are pairwise adjacent, whereas $\{e, \beta_{17}, \beta_{18}\}$ is a triangle; thus the clique number is 3. Colour e with colour 1, every involution with colour 2, and the two elements of each pair $\{h, h^{-1}\}$ with colours 2 and 3. This colouring is proper, so the chromatic number is 3. The set $\{e\}$ is dominating. An independent set with at least two vertices cannot contain e and contains at most one element from each of the 28 pairs. Taking all 15 involutions together with one element from each pair gives $15 + 28 = 43$ vertices, and this can be done in 2^{28} ways.

(iii) For $h \in MI(G)$ the set $\{e, h, h^{-1}\}$ is a subgroup if and only if it contains h^2, that is, $h^2 = h^{-1}$, that is, $o(h) = 3$. By Lemma 4.1(c), $c(G) = c(S_3)\, c(A_4) = 3 \cdot 9 = 27$, so G has 26 elements of order 3 and $(27 - 1)/2 = 13$ triangles that are subgroups. The other $28 - 13 = 15$ triangles consist of elements of order 6. For instance, in $\{e, \beta_{43}, \beta_{44}\}$ with $\beta_{43} = \big((1\,2), (1\,2\,3)\big)$ we have $\beta_{43}^2 = \big(e, (1\,3\,2)\big) \neq \beta_{44}$, so this triangle is not closed under multiplication. ▫

6. Proof of Theorem 1.2

6.1. Order

Proposition 6.1: The elements of G have orders 1, 2, 3 and 6, and there are 1, 15, 26 and 30 of them, respectively. The colour classes of orders 1 and 2 are independent, while those of orders 3 and 6 are not.

Proof: S_3 has 1, 3, 2 elements of orders 1, 2, 3, and A_4 has 1, 3, 8 elements of these orders (Table 2). By Lemma 4.1(a), the number of elements of each order in G is

$$\text{order 1: } 1 \cdot 1 = 1, \qquad \text{order 2: } 1 \cdot 3 + 3 \cdot 1 + 3 \cdot 3 = 15,$$

$$\text{order 3: } 1 \cdot 8 + 2 \cdot 1 + 2 \cdot 8 = 26, \qquad \text{order 6: } 3 \cdot 8 + 2 \cdot 3 = 30.$$

The class of order 1 is $\{e\}$, and the involutions are pairwise non-adjacent by Lemma 3.1. If $o(h) \in \{3,6\}$, then $h^{-1} \neq h$ has the same order as h and is adjacent to it. ▫

With the labelling of Table A.1, the four colour classes are $\{\beta_1\}$, $\{\beta_2, \dots, \beta_{16}\}$, $\{\beta_{17}, \dots, \beta_{42}\}$ and $\{\beta_{43}, \dots, \beta_{72}\}$; see Figure 1.

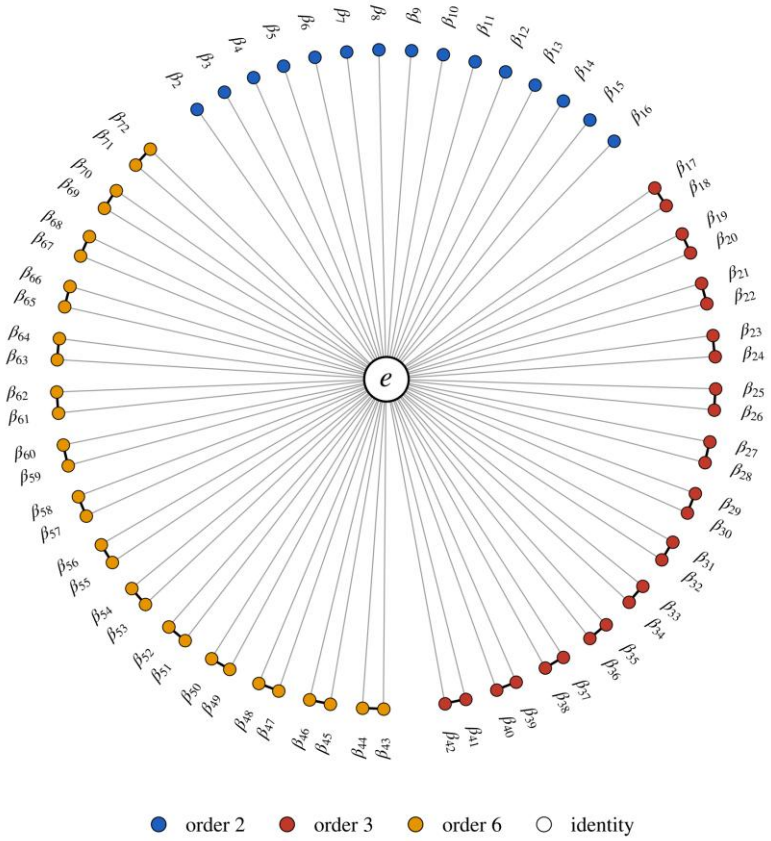


Figure 1: The identity graph $\Omega(G)$, $G = S_3 \times A_4$, coloured by element order. Each vertex carries its label β_i from Table A.1.

6.2. Conjugacy classes

By Lemma 4.2, the $3 \cdot 4 = 12$ conjugacy classes of G are the products of the classes in Table 2. They are numbered $C_1, \dots, C_{12}$ as in Table 3.

Table 3: The twelve conjugacy classes of $G = S_3 \times A_4$.

Class	S_3**-part** $\times$ A_4**-part**	**Size**	**Elements**	**Independent**
C_1	$\{e\} \times \{e\}$	1	β_1	yes
C_2	transpositions $\times \{e\}$	3	$\beta_2, \beta_3, \beta_4$	yes
C_3	$\{e\} \times (V_4 \backslash \{e\})$	3	$\beta_5, \beta_6, \beta_7$	yes
C_4	transpositions $\times (V_4 \backslash \{e\})$	9	$\beta_8, \dots, \beta_{16}$	yes
C_5	3-cycles $\times \{e\}$	2	β_{17}, β_{18}	no

Class	S_3**-part** $\times$ A_4**-part**	**Size**	**Elements**	**Independent**
C_6	$\{e\} \times K_1$	4	$\beta_{19}, \beta_{22}, \beta_{23}, \beta_{26}$	yes
C_7	$\{e\} \times K_2$	4	$\beta_{20}, \beta_{21}, \beta_{24}, \beta_{25}$	yes
C_8	3-cycles $\times K_1$	8	$\beta_{27}, \beta_{30}, \beta_{32}, \beta_{33}, \beta_{35}, \beta_{38}, \beta_{40}, \beta_{41}$	yes
C_9	3-cycles $\times K_2$	8	$\beta_{28}, \beta_{29}, \beta_{31}, \beta_{34}, \beta_{36}, \beta_{37}, \beta_{39}, \beta_{42}$	yes
C_{10}	transpositions $\times K_1$	12	$\beta_{43}, \beta_{46}, \beta_{47}, \beta_{50}, \beta_{51}, \beta_{54}, \beta_{55}, \beta_{58}, \beta_{59}, \beta_{62}, \beta_{63}, \beta_{66}$	yes
C_{11}	transpositions $\times K_2$	12	$\beta_{44}, \beta_{45}, \beta_{48}, \beta_{49}, \beta_{52}, \beta_{53}, \beta_{56}, \beta_{57}, \beta_{60}, \beta_{61}, \beta_{64}, \beta_{65}$	yes
C_{12}	3-cycles $\times (V_4 \backslash \{e\})$	6	$\beta_{67}, \dots, \beta_{72}$	no

Proposition 6.2: Exactly six classes of G are real, namely $C_1, C_2, C_3, C_4, C_5, C_{12}$. The classes C_5 and C_{12} contain edges, and the other ten classes are independent. There are exactly four triangles whose two non-identity vertices lie in one class: $\{e, \beta_{17}, \beta_{18}\}$ in C_5, and $\{e, \beta_{67}, \beta_{68}\}$, $\{e, \beta_{69}, \beta_{70}\}$, $\{e, \beta_{71}, \beta_{72}\}$ in C_{12}.

Proof: Real classes. By Lemma 4.2, $K \times L$ is real exactly when K and L are real. All three classes of S_3 are real, and the real classes of A_4 are $\{e\}$ and $V_4 \backslash \{e\}$ (Table 2). This gives $3 \cdot 2 = 6$ real classes, namely those listed.

Edges. By Lemma 3.3, a class contains an edge exactly when it is real and lies in $MI(G)$. Of the six real classes, $C_1, \dots, C_4$ lie in $SI(G)$ by Lemma 4.2, whereas C_5 and C_{12} consist of elements of orders 3 and 6, respectively.

Triangles. The elements of C_5 and C_{12} form $(2 + 6)/2 = 4$ pairs $\{h, h^{-1}\}$, and by Table A.1 these are the pairs listed. Every other triangle meets two different classes; for example $\{e, \beta_{19}, \beta_{20}\}$ has $\beta_{19} \in C_6$ and $\beta_{20} \in C_7$. ▫

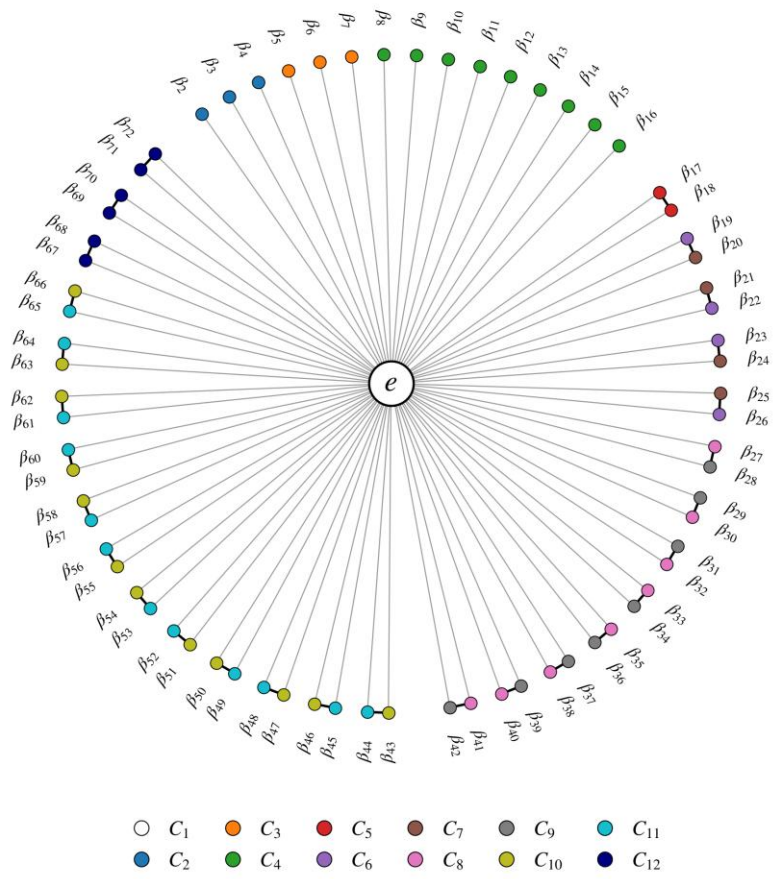


Figure 2: The graph $\Omega(G)$ coloured by conjugacy class. Each of the twelve colours marks one class $C_1, \dots, C_{12}$ of Table 3, and every vertex carries its label β_i.

6.3. Sylow subgroups

Proposition 6.3:

(a) $n_2(G) = 3$ and $n_3(G) = 4$. The Sylow 2-subgroups are $\langle\tau\rangle \times V_4$ with τ a transposition, and the Sylow 3-subgroups are $\langle(1\,2\,3)\rangle \times \langle c\rangle$ with c a 3-cycle of A_4.

(b) Two distinct Sylow 2-subgroups intersect in $\{e\} \times V_4$, of order 4, and two distinct Sylow 3-subgroups intersect in $\langle(1\,2\,3)\rangle \times \{e\}$, of order 3.

(c) $\pi_2(G) = 16$ and $\pi_3(G) = 27$. The 15 non-identity 2-elements form an independent set, the 26 non-identity 3-elements do not, and the 30 elements of order 6 belong to no Sylow subgroup.

Proof: (a) By Lemma 4.3, $n_2(G) = n_2(S_3)\, n_2(A_4) = 3 \cdot 1 = 3$ and $n_3(G) = n_3(S_3)\, n_3(A_4) = 1 \cdot 4 = 4$, and the Sylow subgroups are products of Sylow subgroups of the factors.

(b) For $\tau \neq \tau'$ we get $(\langle\tau\rangle \times V_4) \cap (\langle\tau'\rangle \times V_4) = (\langle\tau\rangle \cap \langle\tau'\rangle) \times V_4 = \{e\} \times V_4$. In the same way, with $A_3 = \langle(1\,2\,3)\rangle$, we get $(A_3 \times \langle c\rangle) \cap (A_3 \times \langle c'\rangle) = A_3 \times \{e\}$ whenever $\langle c\rangle \neq \langle c'\rangle$.

(c) By Lemma 4.1(c), $\pi_2(G) = 4 \cdot 4 = 16$ and $\pi_3(G) = 3 \cdot 9 = 27$. The non-identity 2-elements are the 15 involutions, and these are pairwise non-adjacent. The non-identity 3-elements include the adjacent vertices β_{17} and β_{18}. Finally, $72 - 1 - 15 - 26 = 30$. ▫

Part (b) shows that distinct Sylow p-subgroups share elements, so the colouring is taken by the prime: the 2-elements $\beta_2, \dots, \beta_{16}$, the 3-elements $\beta_{17}, \dots, \beta_{42}$, and the uncoloured elements $\beta_{43}, \dots, \beta_{72}$.

6.4. Centre

By Lemma 4.4, $Z(G) = Z(S_3) \times Z(A_4) = \{e\}$. The colour classes are therefore $\{e\}$ and $G\backslash\{e\}$. The second class contains the edge $\{\beta_{17}, \beta_{18}\}$, so only $\{e\}$ is independent.

6.5. Parity

We regard G as a group of permutations of $3 + 4 = 7$ points, so that $\operatorname{sgn}(a, b) = \operatorname{sgn}(a)\operatorname{sgn}(b)$. Every element of A_4 is even, hence $\operatorname{sgn}(a, b) = \operatorname{sgn}(a)$.

Proposition 6.4: The even elements and the odd elements of G each form a class of 36 vertices. The even class contains 4 self-inverse elements and 16 triangles, and the odd class contains 12 self-inverse elements and 12 triangles. Neither class is independent.

Proof: The two classes have $|A_3| \cdot 12 = 36$ and $3 \cdot 12 = 36$ elements, respectively. An element (a, b) of $SI(G)$ is even exactly when $a = e$; by Lemma 4.1(b) there are $1 \cdot 4 = 4$ such elements and $3 \cdot 4 = 12$ odd ones. Since $\operatorname{sgn}(h^{-1}) = \operatorname{sgn}(h)$, each pair $\{h, h^{-1}\}$ lies inside one class, which gives $(36 - 4)/2 = 16$ and $(36 - 12)/2 = 12$ triangles. The even class contains e, and the odd class contains the adjacent pair $\{\beta_{43}, \beta_{44}\}$. ▫

Proof of Theorem 1.2: The five rows of Table 1 are given by Propositions 6.1, 6.2 and 6.3, Subsection 6.4 and Proposition 6.4. Statements (i), (ii) and (iii) are Propositions 6.2, 6.3 and 6.4. ▫

7. Proof of Theorem 1.3

(i) Normal subgroups. The products of normal subgroups of the two factors,

$$N_1 \times N_2, \qquad N_1 \in \{\{e\}, A_3, S_3\}, \quad N_2 \in \{\{e\}, V_4, A_4\},$$

are nine normal subgroups, of orders 1,4,12,3,12,36,6,24,72. A normal subgroup is a union of conjugacy classes that contains C_1 and whose order divides 72. Going through all unions of the classes in Table 3 shows that there are no further normal subgroups (Section 8).

No normal clique. Suppose that $\{N_1, \dots, N_k\}$ is a normal clique, and let N be the member that contains $g = \big((1\,2), (1\,2\,3)\big)$, an element of order 6. Then N contains $g^3 = \big((1\,2), e\big)$ and $g^4 = \big(e, (1\,2\,3)\big)$, and therefore all their conjugates. The conjugates of $\big((1\,2), e\big)$ are the three elements (τ, e), which generate $S_3 \times \{e\}$. The conjugates of $\big(e, (1\,2\,3)\big)$ are the four elements (e, c) with $c \in K_1$, and they generate $\{e\} \times A_4$, because a subgroup of A_4 that contains two different subgroups of order 3 is A_4 itself. Hence $N = G$, which is excluded.

(ii) By Lemma 3.2, $\Omega(H) \cong \Omega(G)$ holds exactly when $s(H) = 16$. For $H = D_{12} \times C_6$, Lemma 4.1(b) gives $s(H) = s(D_{12})\, s(C_6) = 8 \cdot 2 = 16$, because D_{12} has six reflections and one central involution, and C_6 has one involution. The two groups are not isomorphic, since $Z(G) = \{e\}$ while $Z(D_{12} \times C_6)$ contains $\{e\} \times C_6$. The value of $s(H)$ for all 50 groups H of order 72 was computed with the SmallGroups library [3] (Section 8); only SmallGroup(72, 44) $\cong A_4 \times S_3$ and SmallGroup(72, 48) $\cong C_2 \times C_6 \times S_3 \cong D_{12} \times C_6$ satisfy $s(H) = 16$.

(iii) The dihedral group of order 72 has 36 reflections and one central involution, so $s(D_{72}) = 38 \neq 16$. By Lemma 3.2, $\Omega(G) \ncong \Omega(D_{72})$, and there is no other dihedral group of order 72. ▫

Remark 7.1: Although $\Omega(G) \cong \Omega(D_{12} \times C_6)$, the colourings distinguish the two groups. For example, $D_{12} \times C_6$ has only 8 elements of order 3 and 48 of order 6, against 26 and 30 in G, and it has a unique Sylow 3-subgroup, whereas $n_3(G) = 4$.

8. Computer Verification

Every statement in the paper was checked independently by computer. A Python program lists the 72 elements of G as pairs of permutations and recomputes the element orders, the conjugacy classes, the real classes, the triangles inside classes, the Sylow subgroups and their intersections, the centre, the parity classes, and all unions of conjugacy classes that are subgroups. It confirms every number in Theorems 1.1–1.3, in Tables 1 and 3, and in Table A.1. A SageMath [8] script based on GAP [5] repeats these checks and computes $s(H)$ for the 50 groups of order 72. Both programs are available as supplementary material.

9. Conclusions

This paper has given a complete description of the identity graph of the direct product $S_3 \times A_4$. The graph has 72 vertices and 99 edges and is made up of 15 lines and 28 triangles through the identity, 13 of which are subgroups of the group. For the five colourings, the independent colour classes are the classes of orders 1 and 2, ten of the twelve conjugacy classes, the class of non-identity 2-elements and the class $\{e\}$, while neither parity class is independent. The group has no normal clique, and among the groups of order 72 it shares its identity graph only with $D_{12} \times C_6$. Because every count was reduced to the two factors through the product formulas of Section 4, the same method applies to other direct products, in particular to the family $S_n \times A_m$.

Declarations:

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Appendix A. The elements of $S_3 \times A_4$

Table A.1 lists the 72 elements (a, b), $a \in S_3$, $b \in A_4$, in cycle notation, grouped by order. For each element it gives $o(a)$, $o(b)$ and $o(a, b)$, states whether the element lies in $SI(G)$ or in $MI(G)$, and, for elements of $MI(G)$, gives the label of the inverse.

Table A.1: The complete list of the 72 elements of $S_3 \times A_4$, grouped by order.

Label	Element (a, b)	$o(a)$	$o(b)$	$o(a, b)$	Type	Inverse
β_1	(e, e)	1	1	1	SI	self
β_2	$((12), e)$	2	1	2	SI	self
β_3	$((13), e)$	2	1	2	SI	self
β_4	$((23), e)$	2	1	2	SI	self
β_5	$(e, (12)(34))$	1	2	2	SI	self

Label	**Element** (a, b)	$o(a)$	$o(b)$	$o(a, b)$	**Type**	**Inverse**
β_6	(e, (13)(24))	1	2	2	SI	self
β_7	(e, (14)(23))	1	2	2	SI	self
β_8	((12), (12)(34))	2	2	2	SI	self
β_9	((12), (13)(24))	2	2	2	SI	self
β_{10}	((12), (14)(23))	2	2	2	SI	self
β_{11}	((13), (12)(34))	2	2	2	SI	self
β_{12}	((13), (13)(24))	2	2	2	SI	self
β_{13}	((13), (14)(23))	2	2	2	SI	self
β_{14}	((23), (12)(34))	2	2	2	SI	self
β_{15}	((23), (13)(24))	2	2	2	SI	self
β_{16}	((23), (14)(23))	2	2	2	SI	self
β_{17}	((123), e)	3	1	3	MI	β_{18}
β_{18}	((132), e)	3	1	3	MI	β_{17}
β_{19}	(e, (123))	1	3	3	MI	β_{20}
β_{20}	(e, (132))	1	3	3	MI	β_{19}
β_{21}	(e, (124))	1	3	3	MI	β_{22}
β_{22}	(e, (142))	1	3	3	MI	β_{21}
β_{23}	(e, (134))	1	3	3	MI	β_{24}
β_{24}	(e, (143))	1	3	3	MI	β_{23}
β_{25}	(e, (234))	1	3	3	MI	β_{26}
β_{26}	(e, (243))	1	3	3	MI	β_{25}
β_{27}	((123), (123))	3	3	3	MI	β_{28}
β_{28}	((132), (132))	3	3	3	MI	β_{27}
β_{29}	((123), (132))	3	3	3	MI	β_{30}
β_{30}	((132), (123))	3	3	3	MI	β_{29}

Label	**Element** (a, b)	$o(a)$	$o(b)$	$o(a, b)$	**Type**	**Inverse**
β_{31}	((123), (124))	3	3	3	MI	β_{32}
β_{32}	((132), (142))	3	3	3	MI	β_{31}
β_{33}	((123), (142))	3	3	3	MI	β_{34}
β_{34}	((132), (124))	3	3	3	MI	β_{33}
β_{35}	((123), (134))	3	3	3	MI	β_{36}
β_{36}	((132), (143))	3	3	3	MI	β_{35}
β_{37}	((123), (143))	3	3	3	MI	β_{38}
β_{38}	((132), (134))	3	3	3	MI	β_{37}
β_{39}	((123), (234))	3	3	3	MI	β_{40}
β_{40}	((132), (243))	3	3	3	MI	β_{39}
β_{41}	((123), (243))	3	3	3	MI	β_{42}
β_{42}	((132), (234))	3	3	3	MI	β_{41}
β_{43}	((12), (123))	2	3	6	MI	β_{44}
β_{44}	((12), (132))	2	3	6	MI	β_{43}
β_{45}	((12), (124))	2	3	6	MI	β_{46}
β_{46}	((12), (142))	2	3	6	MI	β_{45}
β_{47}	((12), (134))	2	3	6	MI	β_{48}
β_{48}	((12), (143))	2	3	6	MI	β_{47}
β_{49}	((12), (234))	2	3	6	MI	β_{50}
β_{50}	((12), (243))	2	3	6	MI	β_{49}
β_{51}	((13), (123))	2	3	6	MI	β_{52}
β_{52}	((13), (132))	2	3	6	MI	β_{51}
β_{53}	((13), (124))	2	3	6	MI	β_{54}
β_{54}	((13), (142))	2	3	6	MI	β_{53}
β_{55}	((13), (134))	2	3	6	MI	β_{56}

Label	Element (a, b)	$o(a)$	$o(b)$	$o(a, b)$	Type	Inverse
β_{56}	((13), (143))	2	3	6	MI	β_{55}
β_{57}	((13), (234))	2	3	6	MI	β_{58}
β_{58}	((13), (243))	2	3	6	MI	β_{57}
β_{59}	((23), (123))	2	3	6	MI	β_{60}
β_{60}	((23), (132))	2	3	6	MI	β_{59}
β_{61}	((23), (124))	2	3	6	MI	β_{62}
β_{62}	((23), (142))	2	3	6	MI	β_{61}
β_{63}	((23), (134))	2	3	6	MI	β_{64}
β_{64}	((23), (143))	2	3	6	MI	β_{63}
β_{65}	((23), (234))	2	3	6	MI	β_{66}
β_{66}	((23), (243))	2	3	6	MI	β_{65}
β_{67}	((123), (12)(34))	3	2	6	MI	β_{68}
β_{68}	((132), (12)(34))	3	2	6	MI	β_{67}
β_{69}	((123), (13)(24))	3	2	6	MI	β_{70}
β_{70}	((132), (13)(24))	3	2	6	MI	β_{69}
β_{71}	((123), (14)(23))	3	2	6	MI	β_{72}
β_{72}	((132), (14)(23))	3	2	6	MI	β_{71}

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