Lyapunov's Function for Random Dynamical Systems and Pullback Attractors

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Math. Sub Class :93D30 ,34C35

Abstract. In this paper we study the Lyapunov function for random dynamical systems , where some new properties are proved ,Lyapunov stability theorem ,Rate of pullback convergence.

Key words: random dynamical system ,pullback attractor and Lyapunov function for random dynamical system.

1.Introduction

The concept of random dynamical systems is a comparatively recent development combining ideas and methods from the well developed areas of probability theory and dynamical systems. Due to our inaccurate knowledge of the particular model or due to computational or theoretical limitations (lack of sufficient computational power, in- efficient algorithms or insufficiently developed mathematical and physical theory, for example), the mathematical models never correspond exactly to the phenomenon they are meant to model. Moreover, when considering practical systems we cannot avoid ei- ther external noise or inaccuracy errors in measurements, so every realistic mathematical model should allow for small errors along orbits. To be able to cope with unavoidable uncertainty about the ”correct” parameter values, observed initial states and even the specific mathematical formulation involved, we let randomness be embedded within the model. Therefore, random dynamical systems arise naturally in the modeling of many phenomena in physics, biology, economics, climatology, etc.

The concept of random dynamical systems was mainly developed by Arnold [1] and his ”Bremen group”, based on the research of Baxendale [2], Bismut [3], Elworthy [4], Gihman and Skorohod [5], Ikeda and Watanabe [6] and Kunita [7] on two-parameter stochastic flows generated by stochastic differential equations.

Lyapunov's first method was, however, filled with new life in 1968 when Oseledets [10] proved his celebrated multiplicative ergodic theorem. For (random) dynamical systems under an invariant measure this theorem establishes the existence of Lyapunov exponents as limits and can be used to conclude nonlinear stability from linear stability. A systematic account of the theory of nonlinear random dynamical systems based on Lyapunov's first method through the multiplicative ergodic theorem is given by Arnold [1]. In contrast to his first method, Lyapunov's second method turned out to be very successful from the beginning, in particular in numerous applied problems. Early systematic accounts in the West were given by the Springer Grundlehren volumes of Hahn [9] in 1967 and of Bhatia and Szegö [8] (developing Lyapunov's second method for dynamical systems) in 1970, both of which are still classical references.

D. T. Son (2009)[11] studied the Lyapunov exponents for random dynamical systems. X. Yingchao (2010)[12] used the theory of random dynamical systems and stochastic analysis to research the existence of random attractors and also stochastic bifurcation behavior for stochastic Duffing-van der Pol equation with jumps under some assumptions. I.J.Kadhim and A.H. Khalil(2016)[13] they define the random dynamical system and random sets in uniform space are and proved some necessary properties of these two concepts. Also they sudy the expansivity of uniform random operator.

The structure of this paper is as follows: In Section 2 we recall same basic definition and facts about random dynamical, study the definition of Lyapunov function for random dynamical systems and theorem Lyapunov stability . In Section 3 we will study Lyapunov function for pullback attractor , existence of a pullback absorbing neighborhood system and theorem (Rate of pullback convergence).

2. Lyapunov Functions for Random Dynamical Systems

Definition2.1. A closed random set $M:\Omega \longrightarrow P(x)$is said to be a semi-weak attractor ,if $\forall x \in M(\omega)\exists$a Tempered random variable $\delta_x:\Omega \longrightarrow R^+$and $y \in S(x,\delta_x(\omega)$,there is a sequence $\{t_n\}$in $\mathbb{R}$, $t_n \longrightarrow +\infty \ni d(\varphi(t_n,\theta_{t_n}\omega)x, M(\omega)) \rightarrow 0ast \rightarrow +\infty$

i. a semi –attractor , if

$x \in M, \exists$ tempered random variable $\delta_x \ni S(x,\delta_x(\omega), d\left(\varphi\left(t,\theta_{t_n}\omega\right)x, M(\omega)\right)) \rightarrow 0$as $t \rightarrow +\infty$

ii. a weak attractor , if there is a tempered random variable δ and for each $y \in S\big(M(\omega), \delta(\omega)\big)$, there is a sequence $\{t_n\}$ in $\mathbb{R}$, $t_n \longrightarrow +\infty$ such that $d(\varphi\left(t_n,\theta_{t_n}\omega\right)x, M(\omega) \rightarrow 0$

iii. an attractor , if there is a tempered random variable δ such that for each $y \in S\big(M(\omega), \delta(\omega)\big)$, $d(\varphi(t,\theta_t\omega)x, M(\omega)) \rightarrow 0$ as $t \rightarrow +\infty$

iv. a uniform attractor , if there is a tempered random variable δ such that for each $\varepsilon > 0$, there is T = T (ε) > 0 such that $\{\varphi(t,\theta_t\omega)x: t \in [T,+\infty)\} \subset S(M,\varepsilon)$ for each $x \in S[M,\delta]$

an equi attractor if it is an attractor and if there is $\lambda > 0$ such that for each ε , $0 < \varepsilon < \lambda$ and $T > 0$, there exists tempered random variable δ with $\{\varphi(t,\theta_t\omega)x : t \in [0,T]\} \cap S(M(w),\delta) = \emptyset$ whenever $\varepsilon \leq d(x,M) \leq \lambda$

Definition2.2 (Lyapunov Functions) [1] . Let φ be a random dynamical system in $\mathbb{R}^d$ and A be a random compact set which is invariant under φ. A Function $V:\Omega \times \mathbb{R}^d \rightarrow R^+$ is called Lyapunov Functions for A (under φ) if it is has the following properties :

i. $\omega \mapsto V(\omega, x)$is measurable for each $x \in \mathbb{R}^d$, and $x \longmapsto V(\omega, x)$ is continuous for each $\omega \in \Omega$

ii. V is uniformly unbounded , i.e., $lim_{\|x\| \longmapsto \infty} V(\omega, x) = \infty$ for all ω.

iii. V is positive-definite, i.e., $V(\omega, x) = 0$ for $x \in A(\omega)$,and $V(\omega, x) > 0$ for $x \notin A(\omega)$.

iv. V is strictly decreasing along orbits of φ i.e, $V\left(\theta_t \omega, \varphi(t, \omega, x)\right) V(\omega, x)$ for all $t > 0$ and $x \notin A(\omega)$.

Definition2.3 The derivative of the function $V: \Omega \times \mathbb{R}^d \longmapsto R^+$ along the parametric vector $X(t) = (x_{1\,(t),} x_{2\,(t),} \dots, x_{d\,(t)}$) is defined by

$$\dot{V}\left(\omega, X(t)\right) = \frac{d}{dt} V\left(\omega, X(t)\right) = \sum_{i=1}^{d} \frac{\partial v\left(\omega, x(t)\right)}{\partial x_i} \frac{dx_i}{dt}, \omega \in \Omega. \qquad (1)$$

Theorem2.4 (Lyapunov stability theorem)[1]

Let $x = X(t), x \in S \subseteq \mathbb{R}^d \longmapsto R^+$ has critical point at the origin. If there is function $V: \Omega \,x\mathbb{R}^d \longmapsto R^+$ such that

i. The partial derivative $\frac{\partial v\left(\omega, x(t)\right)}{\partial x_i}$, i=1, 2,.,.,d exist and continuous .

ii. V is positive –definite .

iii. $\dot{V}$ is semi-positive –definite .

Then the origin is stable critical point for the system. If (iii) above replaces by a stronger condition (iii*) $\dot{V}$ is negative –definite.

Then the origin is asymptotical stabile critical point for the systems. If the function that satisfies the hypothesis (i), (ii) and (iii) from the above theorem is called weak Lyapunov function and if hypothesis (iii) is replaced by (iii*), then $v(x, y)$ is called strong Lyapunov function**.**

In the following we shall characterize the asymptotically random set in terms the lyapunov function . to this end we first sate and prove the following lemma.

Lemma2.5 let the phase space X be arbitrary and $K \subset X$. Let $V: \Omega \times K \longmapsto R$ be any continuous function defined on K such that $V\left(\theta_t \omega, \varphi(t, \omega, x)\right) < V(\omega, z)$ for all $t > 0$ and $x \notin K(\omega)$. Whenever

$\varphi([0,t],\omega,x) \subset K(\omega), t \geq 0$.Then if some $x, \varphi(R^+,\omega,x) \subset K(\omega)$, we have $V(\omega,y) = V(\omega,z)$ for every $y,z \in \Lambda^+(x)$.

Proof. Assume that $V\big(\theta_t\omega,\varphi(t,\omega,x)\big) < V(\omega,x)$. there are indeed sequence $\{t_n\}$ and $\{\tau_n\}$ in $\mathbb{R}$ such that $t_n \to \infty$, $\tau_n \to \infty$ and $\varphi(t_n,\omega,x) \to y$, $\varphi(\tau_n,\omega,x) \to z$. We may assume by taking a subsequence that $t_n < \tau_n$ for each n ,Then clearly $V\left(\theta_{t_n}\omega,\varphi(t_n,\omega,x)\right) \geq V\left(\theta_{\tau_n}\omega,\varphi(\tau_n,\omega,x)\right)$ as

$\varphi(\tau_n,\omega,x) = \varphi((\tau_n - t_n) + t_n,\omega,x) = \varphi(\tau_n - t_n),\theta(t_n)\omega,\varphi(t_n,\omega,x)), \tau_n - t_n > 0$, and

$\varphi([0,\tau_n - t_n],\theta(t_n)\omega,\varphi(t_n,\omega,x)) \subset K(\omega)$, Thus proceeding to the limit we have continuity of V ,$V\big(\theta_t\omega,\varphi(t,\omega,y)\big) \geq V(\omega,z)$. This contradicts the original assumption and the limit is proved .

Theorem2.6. A compact random set $M \subset X$ is asymptotically stable if and only if there exists a function $V: \Omega \times N \longrightarrow \mathbb{R}$,where is a neighborhood $N$of M such that

2.6.1 $V(\omega,.): N \to \mathbb{R}$,is continuous $\forall \omega \epsilon \Omega$ and $V(\omega,x): \Omega \longrightarrow \mathbb{R}$ is measurable $\forall x \epsilon N$.

2.6.2 $V(\omega,x) = 0$ if $\forall x \epsilon M$ and $V(\omega,x) > 0$ if $x \notin M, \forall \omega \epsilon \Omega$

2.6.3 $V\big(\theta_t\omega,\varphi(t,\omega,y)\big) < V(\omega,x)$ For all $t > 0$ and $x \notin M(\omega)$ for $x \notin M, t > 0$ and $\varphi([o,t] ,\omega,x) \subset N(\omega) \forall \omega \epsilon \Omega$.

Proof. Assume that a function V as required is given. Choose $\alpha > 0$ such that $S[M,\alpha] \subset N$ and is compact of Let $m = \min\{\Phi(x): x \in H(M,\alpha)\}$,Then by 2.6.2 and continuity of $\Phi, m > 0$. Set $K = \{x \in S[M,\alpha]: \Phi(x) \leq m\}$. Then K is indeed compact and because 2.2.3 k is positively invariant. This establishes that M is stable as K positively invariant neighborhood of. To see that M is an attractor, choose any compact positively invariant neighborhood K of M with $K \subset M$. Then for any $x \epsilon k, \emptyset \neq \Lambda^+ \subset K$, and lemma 2.5 shows that Φ is constant on $\Lambda^+(x) \subset M$. Thus M is attractor and consequently asymptotically stable. Now let M be asymptotically stable and $A(M)$ its region of attractor. For each $x \in A(M)$define

$\varphi(x) = \sup\{\varphi(t,\theta_t\omega)x, M) : t \geq 0\}$

Indeed $\varphi(x)$ is defined for each $x \in A\,(M)$ b, because if $\varrho(\pi(t,x),M) = \alpha,$ Then there is a $T > 0$ with $\pi(T,+\infty),x) \subset S(t,\alpha).$ Thus

$$\varphi(x) = \sup\{\varphi(t,\theta_t\omega)x, M\,) : T \geq t \geq 0\}$$

As $\varrho(\varphi(t,\theta_t\omega)x, M)$ is a continuous function of $t, \varphi(x)$ is define. This $\varphi(x)$ has the properties: $\varphi(x) = 0$ for $x \in M, \varphi(x) \geq 0$ for $x \notin M$ and $\varphi\big(\pi(t,x)\big) \leq \varphi(x)$ for $t \geq 0.$ This is clear when we remember that M is stable and hence positively invariant and that $A\,(M)$ is invariant. Thus if $\varphi(x)$ is defined for any $x \in M$, it is defined for all xt with$t \in R^+$. We further claim that this $\varphi(x)$ is continuous in $M.$ Indeed stability of M implies continuity of $\varphi(x)$ on M . For $x \notin M$, set $\varrho(\pi(t,x),M) = \alpha(>0)$ and choose $\varepsilon, \varepsilon > {}^{\alpha}/_{4}$, such that $S[x,\varepsilon]$ is compact subset of $A\,(M)$ is open , since M is uniform attractor , there is a $T > 0$ such that $S[x,\varepsilon]t \subset S[M, {}^{\alpha}/_{4}]$ for all $t \geq T$. Thus for $y \epsilon S[x,\varepsilon]$ we have

$$\varphi(x) - \varphi(y) = \sup\{\varrho(\varphi(t,\theta_t\omega)x, M) : t \geq 0\} - \sup\{\varrho(\varphi(t,\theta_t\omega)y, M) : t \geq 0\}$$

$$= \sup\{\varrho(\varphi(t,\theta_t\omega)\,x, M) : T \geq t \geq 0\} - \sup\{\varrho(\varphi\,(t,\theta_t\omega)\,y, M) : T \geq t \geq 0\}$$

There fore

$$|\varphi(x) - \varphi(y)| \leq \sup\{\varrho(\varphi(t,\theta_t\omega)x, M) - \varrho(\varphi(t,\theta_t\omega)y, M) : T \geq t \geq 0\}$$
$$\leq \varrho(\varphi(t,\theta_t\omega)x, \varphi(t,\theta_t\omega)y) : T \geq t \geq 0\}.$$

The continuity axiom implies that the right hand side of the above inequality tends to zero as $y \to x$, for T is fixed for $y \epsilon S[x,\varepsilon]$ .The function $\varphi(x)$ is therefore continuous in M . However, the above function may not be strictly decreasing along parts of trajectories in $A\,(M)$ which are not in M and so may not satisfy 2.6.3. Such a function can be obtained by setting

$$V(\omega,x) = \int_0^\infty V(\theta_{\tau_n}\omega, \varphi(\tau_n,\omega,x))\exp(-\tau_n)\,d\tau.$$

That $V(\omega,x)$ is continuous and satisfies 2.6.2 in $A\,(M)$ is clear. To see that $V(\omega,x)$ satisfies 2.6.3, let $x \notin M(\theta_{-t}\omega)$ and $t > 0$. Then indeed $V\big(\theta_t\omega, \varphi(t,\omega,x)\big) < V(\omega,x)$, holds because $v\big(\theta_t\omega, \varphi(\theta_\tau\omega,x)\big) < v(\omega,x)$ holds.

To rule $V\big(\theta_t\omega,\varphi(\theta_\tau\omega,x)\big) < V(\theta_\tau\omega,x) = V(\omega,x), \forall x \in N$, observe that in this case we must have

$V\big(\theta_{t+\tau}\omega,\varphi(t+\tau\ \ ,\omega,x)\big) \equiv V\big(\theta_\tau\omega,\varphi(\tau\ \ ,\omega,x)\big)$ for all $\tau > 0\ , x \in .$

thus in particular, letting $\tau = 0\ , t\ , 2t\ , \ldots$ we get

$\mathrm{V}(\omega\ ,x) = V\big(\theta_{nt}\omega,\varphi(nt\ \ ,\omega,x)\big)\ , n\ = 1,2,3, \ldots$. By asymptotic stability of $M(\theta_{-t}\omega)$ implies that for $x\epsilon A(M),\ \varphi(nt\ \ ,\omega,x) \to 0\ as\ t \to \infty$, $V(\omega,x)$ is continuous. This shows that $V(\omega,x)$ =0. But as $x \notin \mathrm{M}$, we must have $V(\omega,x) > 0$ a contradiction . We have thus proved that $V\big(\theta_t\omega,\varphi(t,\omega,x)\big) < V(\omega,x)$ for$x \notin \mathrm{M}$, and $t > 0.$ the theorem is proved.

3. Lyapunov function for pullback attractor

A Lyapunov function characterizing pullback attraction and pullback attractors for a discrete –time process in $\mathbb{R}^d$ will be constructed here.

Consider a non-autonomous difference equation

$$x_{n+1}(\omega) = f_n(\theta_n\omega, x_n(\omega)) \tag{2}$$

On $\mathbb{R}^d$, where the $f_n: \mathbb{R}^d \to \mathbb{R}^d$, are Lipschitz continuous mapping.

This generates a process $\Phi: z^2 \times \mathbb{R}^d \to \mathbb{R}^d$ through iteration by $\Phi\big(n,n_0,x_0(\omega)\big) = f_{n-1}{}^\circ \ldots {}^\circ f_{n_0}\ (x_0(\omega))$ is Lipschitz continuous for all $n \geq n_0$ the pullback attraction is taken with respect to basin of attraction system, which is define as follows for a process.

Definition 3.1 A basin of attraction system $\mathfrak{D}_{att}$ consist of families $\mathcal{D} = \{D_n: n\ \in z\}$ of non empty pounded random set of $\mathbb{R}^d$ with the property that $\mathcal{D}^{(1)} = \left\{D_n^{(1)}: n \in z\right\} \in \mathfrak{D}_{att}$ if $\mathcal{D}^{(2)} = \left\{D_n^{(2)}: n \in z\right\} \in \mathfrak{D}_{att}$ and $D_n^{(1)} \subset D_n^2$ for all $n\ \in z$.

Although somewhat complicated, the use of such a basin of attraction system allows both non-uniform and local attraction region, which are typical in non-autonomous system, to be handled.

Definition 3.2 A φ-invariant family of nonempty compact $\mathcal{A} = \{A_n : n \in z\}$ is called a pullback attractor with respect to a basin of attraction system $\mathfrak{D}_{att}$ if it is pullback attracting

$$\lim_{j\to\infty} dist(\varphi\left(n,\theta_{n-j}(\omega),D_{n-j}(\omega)\right),A_n(\omega)=0 \quad (3)$$

For all $n \in z$ and all $\mathcal{D}=\{D_n: n \in z\} \in \mathfrak{D}_{att}$.

Obviously $\mathcal{A} \in \mathfrak{D}_{att}$

The construction of the Lyapunov function requires the existence of a pullback absorbing neighborhood family.

Existence of a pullback absorbing neighborhood system

The following lemma shows that three always exists such a pullback absorbing neighborhood system for any given pullback attractor. **Lemma 3.3** if A is a pullback attractor with a basin of attraction system $\mathcal{D}_{att}$ for a process, then there exists a pullback absorbing neighborhood system $B \subset \mathcal{D}_{att}$ of A with random T. φ moreover, B is $\varphi - positive$ invariant

Proof. For each $n_0 \in \mathbb{Z}$ pick $S_{n_0}(\omega) > 0$ such that

$B\left[A_{n_0}(\omega);S_{n_0}(\omega)\right]:=\{x \in R^d: dist\,(X,A_{n_0}\omega) \le S_{n_0}\}$ such that

$\{B\left[A_{n_0}(\omega);S_{n_0}(\omega)\right]:n_0 \in \mathbb{Z}\} \in \mathcal{D}_{att}$, and define

$$B_{n_0}:=\overline{\cup_{j\ge 0}\ \varphi(n_0\,,\theta_{n_0-j}(\omega),B\left[A_{n_0-j}(\omega);S_{n_0-j}(\omega)\right])}$$

Obviously $A_{n_0} \subset \operatorname{int} B\left[A_{n_0}(\omega);S_{n_0}(\omega)\right] \subset B_{n_0}$.

To show positive invariance the two-parameter semi group property will be used in where follows

$$\varphi\left(n_0+1\,,\theta_{n_0}\omega,B_{n_0}\right)$$

$$=\overline{\cup_{j\ge 0}\,\varphi(n_0+1\,,\theta_{n_0}\omega,\varphi\left(\theta_{n_0}\omega,\theta_{n_0-j}\omega,B\left[A_{n_0-j}(\omega);S_{n_0-j}(\omega)\right]\right))}$$

$$=\overline{\cup_{j\ge 0}\,\varphi(n_0+1\,,\theta_{n_0-j}\omega\,,B\left[A_{n_0-j}(\omega);S_{n_0-j}(\omega)\right])}$$

$$=\overline{\cup_{\imath\ge 1}\,\varphi\left(n_0+1\,,\theta_{n_0+1-j}\omega\,,B\left[A_{n_0+1-j}(\omega);S_{n_0+1-j}(\omega)\right]\right)}$$

$$\subseteq \overline{U_{\imath\ge 0}\varphi\left(n_0+1\,,\theta_{n_0+1-j}\omega\,,B\left[A_{n_0+1-j}(\omega);S_{n_0+1-j}(\omega)\right]\right))} = B_{n_0+1}(\omega)$$

so

$\varphi(n_0+1\,,\theta_{n_0}\omega,B_{n_0}(\omega)$. This and the two parameter semi group property again gives

$$\varphi(n_0+1,\theta_{n_0}\omega,B_{n_0}(\omega)=\varphi(n_0+1,\theta_{n_0+1}\omega,\varphi(n_0+1,\theta_{n_0}\omega,B_{n_0}(\omega)$$

$$\subseteq \varphi(n_0+2,\theta_{n_0+1}\omega,B_{n_0+1}(\omega)\subseteq B_{n_0+2}(\omega).$$

The general positive invariance assertion then follows by induction. Now referring to the continuity of $\varphi\left(\theta_{n_0}\omega,\theta_{n_0-j}\omega\right)$ and the Compactness of $B\left[A_{n_0-j}(\omega);S_{n_0-j}(\omega)\right]$the set $\varphi\left(n_0,\theta_{n_0-j}\omega,B\left[A_{n_0-j}(\omega);S_{n_0-j}(\omega)\right]\right)$ is compact for each $j\geq 0$ and $n_0\in\mathbb{Z}$. moreover, by pullback convergence , there exists an

$N=N(n_0,S_{n_0})\in\mathbb{N}$ such that $\varphi(n_0,\theta_{n_0-j}\omega,B\left[A_{n_{0-j}}(\omega),S_{n_0-j}(\omega)\right])$
$\subseteq B[A_{n_0}(\omega);S_{n_0}(\omega)]\subset$B$(\theta_{n_0}\omega)$ for each $j\geq N$. Hence

B$(\theta_{n_0}\omega)=\overline{\cup_{j\geq 0}\varphi(n_0,\theta_{n_0-j}(\omega),B[A_{n_0-j}(\omega),S_{n_0-j}(\omega))}$

$\subseteq B[A_{n_0}(\omega);S_{n_0}(\omega)\cup\overline{\bigcup_{0\leq j<N}\varphi\left(n_0,\theta_{n_0-j}\omega,B\left[A_{n_0-j}(\omega);S_{n_0-j}(\omega)\right]\right)}$

$=\overline{\bigcup_{0\leq j<N}\varphi(n_0,\theta_{n_0-j},B[A_{n_0-j}(\omega);S_{n_0-j}(\omega)]}$), which is compact, so B$(\theta_{n_0}\omega)$ is compact.

To see that $\mathcal{B}$so constructed is pullback absorbing with respect to$\mathfrak{D}_{att}$, let$\mathcal{D}\in\mathfrak{D}_{att}$.

Fix $n_0\in\mathbb{Z}$. since $\mathcal{A}$ is pullback attracting , there exists an $N\left(\mathcal{D},\delta_{n_0},n_0\right)\in\mathbb{N}$ such that
dist$(\varphi\left(n_0,\theta_{n_0-j}\omega,D_{n_0-j}\right),A_{n_0})<\delta(\theta_{n_0}\omega)$ for all $j\geq N\left(\mathcal{D},S_{n_0},n_0\right)$.
but $\varphi(\left(n_0,\theta_{n_0-j}\omega,D_{n_0-j}\right)\subset int\,B[A_{n_0}(\omega),\delta_{n_0}(\omega)]$ and $B[A_{n_0}(\omega),\delta_{n_0}(\omega)]\subset$ B$(\theta_{n_0}\omega)$, so

$\varphi\left(n_0,\theta_{n_0-j}(\omega),D_{n_0-j}\right)\subset int$ B$(\theta_{n_0}\omega)$ for all $j\geq N\left(D,\delta_{n_0},n_0\right)$

Hence $\mathcal{B}$is pullback absorbing as required .

Necessary and sufficient conditions

The main result is the construction of a Lyapunov function that characterizes this pullback attraction

Theorem. **3.4**. Let f_n be uniformly of Lyapunov continuous on $\mathbb{R}^d$ for each $n \in \mathbb{Z}$ and let φ be the process that they generate. In addition, let $\mathcal{A}$ be a φ –invariant family of nonempty compact random sets that is pullback attracting with respect to φ with a basin of attraction system $\mathfrak{D}_{att}$. Then there exists a Lipschitz continuous function $V: \Omega \times \mathbb{R}^d \to \mathbb{R}$, such that

Property 1 (upper bound): for all $n_0 \in z$ and $x_0 \in \mathbb{R}^d$

$$V(n_0, x_0) \leq dist(x_0, A_{n_0}), \quad (4)$$

Property 2 (lower bound): for each $n_0 \in z$ there exists a function $a(n_0, .): \mathbb{R}^+ \to \mathbb{R}^+$ with $a(n_0, 0) = 0$ and $a(n_0, r) > 0$ for all $r > 0$ which is monotonically increasing in r such that

$$a\left(n_0, dist(x_0, A_{n_0})\right) \leq V(\theta_{n_0}\omega, x_0) \text{ For all } x_0 \in \mathbb{R}^d \quad (5)$$

Property 3 (Lipschitz condition): for $n_0 \in z$ and $x_0 , y_0 \in \mathbb{R}^d$,

$$\left|V\left(\theta_{n_0}\omega, x_0\right) - V(\theta_{n_0}\omega, y_0)\right| \leq \|x_0 - y_0\|, \quad (6)$$

Property 4(pullback convergence): for all $n_0 \in z$ and any $\mathcal{D} \in \mathfrak{D}_{att}$

$$limit\ sup_{n\to\infty}\ sup_{zn_0 - n \in D_{n_0 - n}} V\left(\theta_{n_0}\omega, \varphi(n_0, \theta_{n_0 - n}, zn_0 - n\ )\right) = 0 \quad (7)$$

In addition,

Property 5 (forwards convergence) : there exists $\mathcal{N} \in \mathfrak{D}_{att}$, which is positively invariant under φ and consists of nonempty compact random sets $N(\theta_{n_0}\omega)$ with $A_{n_0}(\omega) \subset intA_{n_0}(\omega)$ for each $n_0 \in \mathbb{Z}$ such that

$$V\left(\theta_{n_0+1}\omega, \varphi(n_0 + 1, \theta_{n_0}\omega, x_0\ \right) \leq e^{-1}V\left(\theta_{n_0}\omega, x_0\right) \quad (8)$$

For all $x_0 \in N(\theta_{n_0}\omega)$ and hence

$$V\left(\theta_{n_0+j}\omega, \varphi(j, \theta_{n_0}\omega, x_0\ )\right) \leq e^{-j}V\left(\theta_{n_0}\omega, x_0\right) \text{ for all } x_0 \in N(\theta_{n_0}\omega)$$

$$, j \in \mathbb{N} \quad (9)$$

Proof. The aim is to construct a Lyapunov function $V(\theta_{n_0}\omega, x_0) := sup_{n\in\mathbb{N}}\, e^{-T_{n_0}} dist(\left(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\right)$ for all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$ where $T_{n_0,n} = n + \sum_{j=1}^{n} \alpha_{n_0-j}^{+}$.

With $T_{n_0,0} = 0$. Here $\alpha_n log L_n$ where L_n is the uniform Lipschitz constant f_n on $\mathbb{R}^d$, and$a^+ = (a + |a| \,/\, 2$, that is the positive part of a real number a.

Note 4 $T_{n_0,n} \geq n$ and $T_{n_0,n+m} = T_{n_0,n} + T_{n_0-n,m}$ for $n, m \in \mathbb{N}, n_0 \in \mathbb{Z}$.

Proof 1

Since $e^{-T_{n_0,n}} \leq 1$ for all $n \in \mathbb{N}$ and $dist\left(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\right)$

Is monotonically increasing from $0 \leq dist\left(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\right) \leq 1. dist(x_0, A(\theta_{n_0}\omega)$.

Proof 2

If$x_0 \in A(\theta_{n_0}\omega)$,then

$V(\theta_{n_0}\omega, x_0) = sup_{n\geq 0}\, e^{-T_{n_0-n}} dist(\left(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\right))$ the supremum involves the product of an exponentially quantity bounded below by zero and a bounded increasing function , since the set $\varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)$ are a nested family of compact random sets decreasing to $A(\theta_{n_0}\omega)$ with increasing n. In particular, $dist\left(x_0, A(\theta_{n_0}\omega\right) = \left(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\right)$ for all $n \in \mathbb{N}$. Hence there exists an $N^* = N^*(n_0, x_0) \in \mathbb{N}$ such that

$$\frac{1}{2} dist(x_0, A(\theta_{n_0}\omega) \leq dist(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega) \leq dist(x_0, A(\theta_{n_0}\omega)$$

.For all $n \geq N^*$, but not $n = N^* - 1$. There, from above,

$$V(\theta_{n_0}\omega, x_0) \geq e^{-T_{n_0-N^*}} dist((x_0, \varphi(n_0, \theta_{n_0-N^*}\omega, B(\theta_{n_0-N^*}\omega)) \geq \frac{1}{2} e^{-T, N^*}(x_0, A(\theta_{n_0}\omega))$$

Define

$$N^*(\theta_{n_0}\omega, r) := \sup\left\{N^*(\theta_{n_0}\omega, x_0) : dist\left(x_0, A(\theta_{n_0}\omega)\right) = r\right\}$$

Journal of Iraqi Al-Khwarizmi Society (JIKhS) Volume:3 Issue: Special 2019 pages: 26-41

Now

$N^*(\theta_{n_0}\omega, r) < \infty$ For $x_0 \notin A(\theta_{n_0}\omega)$ with $dist\left(x_0, A(\theta_{n_0}\omega)\right) = r$ and $N^*(\theta_{n_0}\omega, r)$ is non decreasing with $r \to 0$. To see this note that by the triangle rule

$$dist\left(x_0, A(\theta_{n_0}\omega)\right) \leq dist((x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)) + dist(\varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)), A(\theta_{n_0}\omega))$$

.

Also by pullback convergence there exists an $N(n_0, {}^r/_2)$ such that

$dist(\varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)), A(\theta_{n_0}\omega)) < \frac{1}{2}r$. For all $n \in N(n_0, {}^r/_2)$. Hence for $dist\left(x_0, A(\theta_{n_0}\omega)\right) = r$ and $n \geq N(n_0, {}^r/_2)$,

That is ${}^r/_2 \leq dist\left(x_0, \varphi\left(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)\right)\right)$.

Obviously $N^*(\theta_{n_0}\omega, r) \leq N^*(\theta_{n_0}\omega, {}^r/_2)$.

Finally, define $a(n_0, r) := {}^r/_2\, e^{T_{n_0, N^*(n_0, r)}}$. (10)

Note that there is no guarantee here (with further assumption) that $a(n_0, r)$ dose not converge to 0 for fixed $r \neq 0)$ as $n_0 \to \infty$.

Proof 3

$$\left|V(\theta_{n_0}\omega, x_0) - V(\theta_{n_0}\omega, y_0)\right|$$

$$= \left|sup_{n\epsilon N}\, e^{-T_{n_0-n}} dist((x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)) - sup_{n\epsilon N}\, e^{-T_{n_0-n}} dist((y_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega))\right|$$

$$\leq sup_{n\epsilon N}\, e^{-T_{n_0-n}} \left|dist((x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)) - dist((y_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega))\right|$$

$$\leq \sup_{n\epsilon N} e^{-T_{n_0-n}} \|x_0 - y_0\| \leq \|x_0 - y_0\|$$

Since

$|dist(x_0, C) - dist(y_0, C)| \leq \|x_0 - y_0\|$ for any $x_0, y_0 \in \mathbb{R}^d$ and nonempty compact random subset of $C \in \mathbb{R}^d$.

Proof 4

Assume the opposite. Then there exists an $\varepsilon_0 > 0$, a sequence $n_j \to \infty$ in $\mathbb{N}$ and points $x_j \in \varphi(n_0, \theta_{n_0-n}\omega, D(\theta_{n_0-n}\omega)$ such that $V(\theta_{n_0}\omega, x_0) \geq \varepsilon_0$ for all $\varepsilon_0 \in \mathbb{N}$ since $\mathcal{D} \in \mathfrak{D}_{att}$ and $\mathcal{B}$ is pullback absorbing , there exists an $N = N(\mathcal{D}, n_0) \in \mathbb{N}$ such that $\varphi(n_0, \theta_{n_0-n_j}\omega, D(\theta_{n_0-n_j}\omega) \subset B(\theta_{n_0}\omega)$ for all $n_j \geq$N.

Hence, for all j such that $n_j \geq \mathrm{N}$, it holds $x_j \in B(\theta_{n_0}\omega)$, which is a compact random set , so there exists a convergent subsequence $x_j \to x^* \in B(\theta_{n_0}\omega)$. but also

$$x_j \in \overline{\cup_{n_j \geq \mathrm{N}} \varphi(n_0, \theta_{n_0-n}\omega, D(\theta_{n_0-n}\omega)})$$

And $\cap_{n_j, n \geq n_j} \overline{\cup_{n_j \geq \mathrm{N}} \varphi(n_0, \theta_{n_0-n}\omega, D(\theta_{n_0-n}\omega)}) \subset A(\theta_{n_0}\omega)$

And the definition of a pullback attractor. Hence $x^* \in A(\theta_{n_0}\omega)$

And $V(\theta_{n_0}\omega, x^*) = 0$. But V is Lipschitz continuous in its second variable by property 3, so $\epsilon_0 \leq V(\theta_{n_0}\omega, x_j) = \|V(\theta_{n_0}\omega, x_j) - V(\theta_{n_0}\omega, x^*)\| \leq \|x_j - x^*\|$, which contradicts the convergence $x_j \to x^*$. Hence, property 4 must hold.

Proof 5

Define $N_{n_0} := \{ x_0 \in B[B(\theta_{n_0}\omega), 1]: \varphi(n_0 + 1, \theta_{n_0}\omega, x_0) \in B(\theta_{n_0+1}\omega)\}$,

Where $B[B(\theta_{n_0}\omega), 1] = \{x_0 : dist(x_0, B(\theta_{n_0}\omega)) \leq 1\}$ is bounded because $B(\theta_{n_0}\omega)$ is random compact and $\mathbb{R}^d$ is locally compact, so N_{n_0} is bounded. It is also closed, hence compact, since

$\varphi(n_0+1,\theta_{n_0}\omega,.)\ A(\theta_{n_0}\omega)\subset intB(\theta_{n_0}\omega)$ is continuous and $B(\theta_{n_0}\omega)$ is compact. Now and $B(\theta_{n_0+1}\omega)\subset N_{n_0}$ so $A(\theta_{n_0}\omega)\subset intN_{n_0}$. In addition

$\varphi(n_0+1,\theta_{n_0}\omega,Nn_0)\subset B(\theta_{n_0+1}\omega)\subset N_{n_0+1}$, so $\mathcal{N}$ is positive invariant.

It remains to establish the exponential decay inequality (40). This needs the following Lipschitz condition on $\varphi(n_0,\theta_{n_0}\omega,.)\equiv f_{n_0}(.)$: $\|\varphi(n_0,\theta_{n_0}\omega,x_0)-\varphi(n_0,\theta_{n_0}\omega,y_0)\|\leq e^{\alpha_{n_0}}\|x_0-y_0\|$.

For all $x_0,y_0\in D(\theta_{n_0}\ \omega)$ .it follows from this that

$$dist(\varphi(n_0+1,\theta_{n_0}\omega,x_0)-\varphi(n_0+1,\theta_{n_0}\omega,C_{n_0})\ \leq e^{\alpha_{n_0}}dist(x_0,C_{n_0})$$

From the definition of V, $V\left((\theta_{n_0+1}\omega),\varphi(n_0+1,\theta_{n_0}\omega,x_0)\right)=$

$$\sup_{n\geq 0} e^{-T_{n_0+1,n}}\, dist(\varphi(n_0+1,\theta_{n_0}\omega,x_0),(\varphi\left(n_0,\theta_{n_0-n}\omega,B(\theta_{n_0-n}\omega)\right)$$

Since $\varphi(n_0+1,\theta_{n_0}\omega,x_0)\in B(\theta_{n_0+1}\omega)$ when $x_0\in N(\theta_{n_0}\omega)$. Hence re-indexing and then using the two-parameter semi group property and the Lipschitz condition on $\varphi(1,\theta_{n_0}\omega,.)$.

$$V\left(\theta_{n_0+1}\omega\,,\varphi(n_0+1,\theta_{n_0}\omega,x_0)\right)=$$

$$\sup_{j\geq 0} e^{-T_{n_0+1,j+1}}\, dist\,(\varphi(n_0+1,\theta_{n_0}\omega,x_0),(\varphi\left(n_0,\theta_{n_0-j-1}\omega,B(\theta_{n_0-j-1}\omega)\right)$$
$$=\sup_{j\geq 0} e^{-T_{n_0+1,j+1}}\, dist(\varphi(n_0+1,\theta_{n_0}\omega,x_0),(\varphi\left(n_0+1,\theta_{n_0}\omega,\varphi\left(n_0,\theta_{n_0-j}\omega,B(\theta_{n_0-j}\omega)\right)\right)$$
$$sup_{j\geq 0}\ e^{-T_{n_0+1,j+1}}\ e^{\alpha_{n_0}}dist(x_0,\varphi\left(n_0,\theta_{n_0-j}\omega,B(\theta_{n_0-j}\omega)\right))$$
$$T_{n_0+1,j+1}=T_{n_0\ ,j+1}-\alpha_{n_0}{}^{+},V(\theta_{n_0+1}\omega,\varphi(n_0+1,\theta_{n_0}\omega,x_0)$$
$$\leq\sup_{j\geq 0} e^{-T_{n_0+1,j+1}}\, e^{\alpha_{n_0}}\, dist(x_0,\varphi\left(n_0,\theta_{n_0-j}\omega,B(\theta_{n_0-j}\omega)\right))$$
$$=e^{-1}\sup_{j\geq 0} e^{-T_{n_0,j}}\, dist(x_0,\varphi\left(,\theta_{n_0-j}\omega,B(\theta_{n_0-j}\omega)\right))$$

$\leq e^{-1}V(\theta_{n_0}\omega,x_0)$, Which is the desired inequality. Moreover, since $\varphi(1,\theta_{n_0}\omega,x_0)\in\ B(\theta_{n_0+1}\omega)\subset\ N(\theta_{n_0+1}\omega)$,the proof continues inductively to give, $V(\theta_{n_0+j}\omega, \varphi(n_0+1, \theta_{n_0}\omega, x_0) \leq e^{-j} V(\theta_{n_0}\omega, x_0)$, for all $j \in \mathbb{N}$. This completes the proof of theorem 3.4.

Definition. 3.5. A family random sets $\mathcal{D} \in \mathfrak{D}_{att}$ is called past-tempered with respect to A if

$\lim_{j\to\infty} \frac{1}{j} \log^{+} dist\left(\left(\theta_{n_0}\omega\right), A\left(\theta_{n_0-j}\omega\right)\right) = 0$ .For all $n_0 \in \mathbb{Z}$, or equivalently if $\lim_{j\to\infty} e^{-\gamma j}\ dist\left(D\left(\theta_{n_0-j}\omega\right), A\left(\theta_{n_0-j}\omega\right)\right) = 0$ for all $n_0 \in \mathbb{Z}$, $0 < \gamma$.

Proposition3.6. For past-tempered family random sets, $D \subset \mathcal{N}$ it follows that

$$\lim_{j\to\infty} dist\varphi\left(n_0, \theta_{n_0-j}\omega, D\left(\theta_{n_0-j}\omega\right), A\left(\theta_{n_0-j}\omega\right)\right) = 0$$

Proof

$V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-j}\omega, x\left(\theta_{n_0-j}\omega\right)\right) \leq e^{-j} dist\left(D\left(\theta_{n_0-j}\omega\right), A\left(\theta_{n_0-j}\omega\right)\right) \to 0$ as $j \to \infty$. Hence

$$a(\theta_{n_0}\omega, dist\varphi\left(n_0, \theta_{n_0-j}\omega, x\left(\theta_{n_0-j}\omega\right), A\left(\theta_{n_0-j}\omega\right)\right)$$
$$\leq e^{-j} dist\left(D\left(\theta_{n_0-j}\omega\right), A\left(\theta_{n_0-j}\omega\right)\right) \to 0 \text{ as } j \to \infty.$$

Since n_0 is fixed in the lower expression, this implies the pullback convergence

$$\lim_{j\to\infty} dist\varphi\left(n_0, \theta_{n_0-j}\omega, D\left(\theta_{n_0-j}\omega\right), A\left(\theta_{n_0}\omega\right)\right) = 0$$

Theorem3.7. (Rate of pullback convergence)

If $\mathcal{B}$ is pullback absorbing neighborhood system, then for all $n_0 \in \mathbb{Z}, n \in \mathbb{N}$ and $\mathcal{D} \in \mathfrak{D}_{att}$ there exists an $N(\mathcal{D}, n_0, n) \in \mathbb{N}$ such that $V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z\left(\theta_{n_0-m}\omega\right)\right)$

$$\leq e^{-T_{n_0,n}} dist\left(B\left(\theta_{n_0}\omega\right), A\left(\theta_{n_0}\omega\right)\right)$$

Proof

$$\varphi\left(n_0, \theta_{n_0-n-m}\omega, D\left(\theta_{n_0-n-m}\omega\right)\right)$$
$$= \varphi\left(n_0, \theta_{n_0-n}\omega, \varphi\left(n_0-n, \theta_{n_0-n-m}\omega, D\left(\theta_{n_0-n-m}\omega\right)\right)\right)$$

$$\subset \varphi\left(n_0, \theta_{n_0-n}\omega, B\left(\theta_{n_0-n}\omega\right)\right)$$
$$= \varphi\left(n_0, \theta_{n_0-i}\omega, \varphi\left(n-i, \theta_{n_0-n}\omega, B\left(\theta_{n_0-n}\omega\right)\right)\right)$$
$\subset \varphi\left(n_0, \theta_{n_0-i}\omega, B\left(\theta_{n_0-i}\omega\right)\right)$ .For every $m \geq N, n \geq i \geq 0$

Where the positive invariance of $\mathcal{B}$ was used the last line. Hence

$\varphi\left(n_0 - n, \theta_{n_0-n-m}\omega, D\left(\theta_{n_0-n-m}\omega\right)\right) \subset \varphi\left(n_0 - n, \theta_{n_0-n-m}\omega, D\left(\theta_{n_0-n-m}\omega\right)\right).$

For every $m \geq N(D, n_0, n)$ and $n \geq i \geq 0$ or equivalently

$\varphi\left(n_0 - n, \theta_{n_0-m}\omega, D\left(\theta_{n_0-m}\omega\right)\right) \subseteq \varphi\left(n_0 - n, \theta_{n_0-i}\omega, B\left(\theta_{n_0-i}\omega\right)\right)$ .For every $m \geq n + N(D, n_0, n)$ and $n \geq i \geq 0$.

This means that for any $Z\left(\theta_{n_0-m}\omega\right) \in D\left(\theta_{n_0-m}\omega\right)$ the supremum in

$$V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z\left(\theta_{n_0-m}\omega\right)\right) =$$
$$e^{-T_{n_0,i}} dist\left(\varphi(\theta_{n_0}\omega), \theta_{n_0-m}\omega, Z\left(\theta_{n_0-m}\omega\right)\right), \varphi\left(n_0 - n, \theta_{n_0-i}\omega, B\left(\theta_{n_0-i}\omega\right)\right)$$

Need only be consider over $i \geq n$. Hence

$$V\left(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z\left(\theta_{n_0-m}\omega\right)\right)\right.$$
$$= \sup_{i\geq 0} e^{-T_{n_0,i}} dist\varphi\left(n_0, \theta_{n_0-m}\omega, Z\left(\theta_{n_0-m}\omega\right)\right), \varphi\left(n_0, \theta_{n_0-i}\omega, B\left(\theta_{n_0-i}\omega\right)\right)\Big)$$
$$\leq e^{-T_{n_0,n}} \sup_{j\geq 0} e^{-T_{n_0,n,j}} dist\varphi\left(n_0, \left(\theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right), A\left(\theta_{n_0}\omega\right)\right)$$
$$\leq e^{-T_{n_0,n}} dist(B\left(\theta_{n_0}\omega\right), A\left(\theta_{n_0}\omega\right))$$

Since $A\left(\theta_{n_0}\omega\right) \subseteq \varphi\left(n_0, \theta_{n_0-i-j}\omega, B\left(\theta_{n_0-i-j}\omega\right)\right)$ and

$\varphi\left(n_0, \theta_{n_0-m}\omega, Z\left(\theta_{n_0-m}\omega\right)\right) \in B\left(\theta_{n_0}\omega\right)$

Thus $V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z\left(\theta_{n_0-m}\omega\right)\right) \leq e^{-T_{n_0 n}} dist(B\left(\theta_{n_0}\omega\right), A\left(\theta_{n_0}\omega\right)).$

For every $Z\left(\theta_{n_0-m}\omega\right) \in D\left(\theta_{n_0-m}\omega\right), m \geq n + N(D, n_0, n)$ and $n \geq 0$

Corollary3.8. We can be assumed that the mapping $n \longmapsto n + N(D, n_0, n)$

If monotonic increasing in n (by taking a large $N(D, n_0, n)$ if necessary, and is hence invertible. Let the inverse of $n = M(m) = N(D, n_0, n)$. Then

$$V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z\left(\theta_{n_0-m}\omega\right)\right)$$

$$\leq e^{-T\, n_0, M(m)} dist(B\left(\theta_{n_0}\omega\right), A\left(\theta_{n_0}\omega\right))$$

For every $m \geq N(D, n_0, 0) \geq 0$.

Usually $N(D, n_0, 0) > 0$. This expression can be hold for every $m \geq 0$ by replacing $M(m)$ by $M^*(m)$ defined for every $m \geq 0$ and introducing a constant $K_{D,n_0} \geq 1$ to account for the behavior over the finite time set$m \geq N(D, n_0, 0) \geq 0$,for every $m \geq 0$,thisgives

$$V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z\left(\theta_{n_0-m}\omega\right)\right) \leq K_{D,n_0} dist\left(B\left(\theta_{n_0}\omega\right), A\left(\theta_{n_0}\omega\right)\right).$$

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