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On fuzzy soft normed space

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Abstract :

In this paper , we have introduced the definition of fuzzy soft normed space and obtained some new properties of these space by studying the open and closed balls. Moreover , we studied the continuity and the convergences in fuzzy soft normed space .

Keywords: fuzzy soft norme , fuzzy soft set . fuzzy soft continuity

1. Introduction:

In 2002 , Maji et.al gave a new concept called fuzzy soft set , After the rontier work of Maji, many investigator have extended this concept in various branches of mathematics and Kharal and Ahmad in [2] introduced new theories like new properties of fuzzy soft set and then in [3] defined the concept of mapping on fuzzy soft classes and studies of fuzzy soft in topological introduced by Tanay and Kandemir [4].Mahanta and Das [5] continued studier .In all of the above –mentioned works , the researchers used a fuzzy soft vector space or soft vector space ,while in this worke we used a vector space . In this work we introduce Fuzzy soft normed space and discussed the continuity and convergence and bounded

2.Preliminaries

In this work we use the simples X , E , P(x) to denote for an initial universe ,a set of parameters and the collection of all subsets of X , respectively .

Definition (2.1): [1] A fuzzy set A in X is characterized by a function with domain as X and value in I. The collection of all fuzzy sets in X is denoted by I^X

Definition (2.2) [15] : Let X be a universe set and E be a set of parameters , $P(X)$ the power set of X and $A \subseteq E$ .A pair (F,A) is called soft set over X with recepect to A and F is a mapping given by $F: A \rightarrow P(X)$, $(F,A) = \{F(e) \in P(X): e \in A\}$.

Definition 2.3 [1] : Let A be a subset of E . A pair (F ,A) is called a fuzzy soft set over (X, E) ,if F: $A \rightarrow I^X$ is a mapping from A into I^X .The collection of all fuzzy soft sets over (X , E) is denoted by F(X , E)

Definition (2.4)[1]: A Fuzzy soft set (F,A) over (X,E) is said to be absolute fuzzy soft set , if for all e $\in$ A , F(e) is a fuzzy universal set $\tilde{1}$over X and denoted it by $\tilde{E}$

Definition(2.5)[1]: A fuzzy soft set (F,A) over (X , E) is said to be null fuzzy soft set ,if for all e $\in$ A, F(e) is the null fuzzy set $\tilde{0}$ over X .we denoted it by $\tilde{\Phi}$

Definition(2.6)[41] Let X be a non-empty set, $*$ be a continuous t-norm on I = [0, 1]. A function $N : X \times (0,\infty) \rightarrow [0,1]$ is called a fuzzy norm function on X if it satisfies the following axioms: $for\ all\ x,y\ \in X,\ t,s > 0;$

1) $N(x,t) > 0.$
2) $N(x,t) = 1 \Leftrightarrow x = 0.$
3) $N(\alpha x,t) = N\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{F}/\{0\}$.
4) $N(x,t) * N(y,s) \le N(x+y,t+s).$
5) $N(x,.) : (0,\infty) \rightarrow [0,1]$ is continuous.
6) $\lim_{t\rightarrow\infty} N(x,t) = 1.$

$(X,N,*)$ is called a fuzzy normed space.

Definition(2.7) : Let X be a vector space .Then a mapping $\|.\| : X \rightarrow R(E)^*$ is said to be a soft norm on X if $\|.\|$ satisfies the following conditions :

1) $\|x\| \ge 0\ for\ all\ x \in X$

2) $\|x\| = 0 \leftrightarrow x = 0$

3) $\|r\,x\| = |r|\|x\|\ for\ all\ x \in X\ and\ for\ every\ soft\ scalar\ r$

4) $\|x+y\| \le \|x\| + \|y\|\ for\ all\ x,y\ \in X$

The vector space X with a soft norm $\|.\|$ on X is said to be soft normed space and denoted by (X , $\|.\|$)

3.Main result

Definition(3.1) :Let X be a vector space over the scalar filed K, suppose $*$ is continuous t-norm , and. A fuzzy sub set Γ on X x $(0,\infty)$ is called fuzzy soft norm on X if and only if for $x_e, y_{e'} \in$ X and $k \in K$ the following condition hold

1) $E(x_e, t) = 0\ \forall t \le 0$

2) $E(x_e, t) = 1\ \forall t \ge 0$ if and only if $x_e = \theta_0$

3) $E(k\,x_e, t) = E(x_e, \frac{t}{|k|})$ if $k \neq 0\ \forall\ t > 0$

4) $E(x_e \oplus x_{e'}, t \oplus s) \ge E(x_e, t) * E(y_{e'}, s)\ \forall\ t,s > 0\ and\ x_e, y_e \in X$

5) $E(x_e, .)$ is continuous function and $\lim_{t\rightarrow\infty} E(x_e, t) = 1$

The triple (X, E , $\|.\|$) will be refered to a fuzzy soft normed space

Definition(3.2): let (X , E, $\|.\|$) be a fuzzy soft normed space and $t > 0$ we define an open ball , a closed ball and sphere with center at x_e and radius $\propto$ as follows

$B(x_{e1}, r, t) = \{y_{e2} \in X: E(x_{e1} - y_{e2}, t) > 1 - r\}$

$\bar{B}(x_{e1}, r, t) = \{y_{e2} \in X: E(x_{e1} - y_{e2}, t) \geq 1 - r\}$

$S(x_{e1}, r, t) = \{y_{e2} \in X: E(x_{e1} - y_{e2}, t) = 1 - r\}$

SFS(B(x_{e1} , r , t)) , SFS(B(x_{e1} , r , t)) and SFS(S(x_{e1} , r , t)) are called fuzzy soft open ball, fussy soft closed ball , fuzzy soft sphere respectively with center x_e and radius r

Definition(3.3):A mapping $\Delta: X \times X \times (0,\infty) \to (0,1)$ is said to be fuzzy soft metric on X if Δ satisfies the following condition

1) $\Delta(x_{e1}, y_{e2}, t) = 0$ for all $t \leq 0$

2) $\Delta(x_{e1}, y_{e2}, t) = 1$ for all $t \geq 0$ if and only if $x_{e1} = y_{e2}$

3) $\Delta(x_{e1}, y_{e2}, t) = \Delta(y_{e2}, x_{e1}, t)$

4) $\Delta(x_{e1}, z_{e3}, s \oplus t) \geq \Delta(x_{e1}, y_{e2}, s) * \Delta(y_{e2}, z_{e3}, t) \quad \forall\, t, s > 0$

5) $\Delta(x_{e1}, y_{e2}, .) : (0,\infty) \to (0,1)$ is continuous .

X with a fuzzy soft metric Δ is called a fuzzy soft metric space and denoted by $(X, \Delta, *)$

Definition(3.4) : Let { x_{ej}^{n} } be a sequence of vectors in a fuzzy soft normed space $(X, E, \|.\|)$.Then the sequence convergence to x_{ej}^{0} with respect to fuzzy soft norm.

If $(x_{ej}^{n} - x_{ej}^{0}, t) \geq 1 - \alpha$ for every $n \geq n_0$ and $\alpha \in (0,1]$ where n_0 is positive integer and $t > 0$

Or $\lim_{n\to\infty} E(x_{ej}^{n} - x_{ej}^{0}, t) = 1 \;\; as\, t \to \infty$

Similarly if $\lim_{n\to\infty} \Delta(x_{ej}^{n} - x_{ej}^{0}, t) = 1$ as $t \to \infty$,then { x_{ej}^{n} } is convergent sequence in fuzzy soft metric space $(X, \Delta, *)$

Definition(3.5): A sequence { x_{ej}^{n} } in a fuzzy soft normed space $(X, E, \|.\|)$ is said to be a Cauchy sequence with respect to the fuzzy soft norm if

$E(x_{ej}^{n} - x_{ej}^{m}, t) \geq 1 - \alpha$ for every $n, m \geq n_0$ and $\alpha \in (0,1]$ where n_0 is positive integer and $t > 0$

Or $\lim_{n,m\to\infty} E\,(x_{ej}^{n} - x_{ej}^{m}, t) = 1$ as $t \to \infty$

Similarly if $\lim_{n\to\infty} \Delta\,(x_{ej}^{n} - x_{ej}^{0}, t) = 1$ as $t \to \infty$ then $\{ x_{ej}^{n} \}$ is a Cauchy sequence in fuzzy soft metric space $(X, \Delta, *)$.

Definition(3.6): let $(X, E, \|.\|)$ be a fuzzy soft normed space .Then $(X, E, \|.\|)$ is said to be complete if every Cauchy sequence in X converge.

Definition(3.7):A Complete fuzzy soft normed space is called a fuzzy soft banach space.

Definition(3.8): let $\{x_{ej}^{n}\}$a sequence in a fuzzy soft metric space $(X, \Delta, *)$. Then the sequence $\{x_{ej}^{n}\}$is said to be a bounded sequence with respect to the fuzzy soft metric Δ if $\|x_{ej}^{n} - x_{ej}^{m}\|_{\alpha} \leq M$

By definition $\|x_{ej}^{n} - x_{ej}^{m}\|_{\alpha}$ = inf {t ; $\Delta\,(x_{ej}^{n}, x_{ej}^{m}, t) \geq \alpha$, $\alpha \in (0,1]$ }

That is $\{x_{ej}^{n}\}$ is said to be bounded if there exist a positive integer N depending on M such that $\Delta\,(x_{ej}^{n}, x_{ej}^{m}, t) \geq \alpha$, $\forall n, m \geq N(M)$.

Theorem(3.9) : Every convergent sequence is Cauchy sequence.

Proof : Let$\{x_{ej}^{n}\}$ be a sequence in a fuzzy soft normed space $(X, E, \|.\|)$. Consider $\{x_{ej}^{n}\}$ converges to x_{ej}^{0} .

Then we have $E\left(x_{ej}^{n}, -x_{ej}^{0}, t\right) \geq 1 - \alpha$ for every $n \geq n_0$ and $\alpha \in (0,1]$ $where\, n_0 \in N$ and $t > 0$

Therefore

$$E\left(x_{ej}^{n} - x_{ej}^{m}, t\right) = E\left(x_{ej}^{n} - x_{ej}^{m} \oplus x_{ej}^{0} - x_{ej}^{0}, t\right)$$

$$= E\left((x_{ej}^{n} - x_{ej}^{0}) \oplus (x_{ej}^{m} - x_{ej}^{0}), t\right)$$

$$\geq E\left(x_{ej}^{n} - x_{ej}^{0}, \tfrac{t}{2}\right) * E\left(x_{ej}^{m} - x_{ej}^{0}, \tfrac{t}{2}\right)$$

$$\geq (1-\alpha) * (1-\alpha)$$

$$= \min\{1-\alpha, 1-\alpha\}$$

$$= 1-\alpha$$

$E\left(x_{ej}^{n} - x_{ej}^{m}, t\right) \geq 1 - \alpha$ for every n ,m$\geq n_0$ and $\alpha \in (0,1]$

Thus $\{x_{ej}^{n}\}$ is a Cauchy sequence >

Theorem(3.10): limit of a sequence in fuzzy soft normed space if exist is unique.

Proof :

Let $\{x_{ej}^{n}\}$ be a sequence in a fuzzy soft normed space $(X, E, \|.\|)$.

Such that $\lim_{n\to\infty} \mathrm{E}(x_{ej}^{n} - x_e, \mathrm{t}) = 1$

$\mathrm{Lim}_{n\to\infty} \mathrm{E}(x_{ej}^{n} - x_{e^\backslash}, \mathrm{t}) = 1$, are two limits of sequence $\{x_{ej}^{n}\}$.

Then by definition there exist positive integers n_1, n_2 such that

$\mathrm{E}(x_{ej}^{n} - x_e, \mathrm{t}) \geq 1 - \alpha$ for every $n \geq n_1$ and $\alpha \in (0\,, 1\,]$

$\mathrm{E}(x_{ej}^{n} - x_{e^\backslash}, \mathrm{t}) \geq 1 - \alpha$ for every $n \geq n_2$ and $\alpha \in (0\,, 1\,]$

Choose $n \geq n_0$, n_0 = min$\{n_1, n_2\}$

$\mathrm{E}\,(x_e - x_{e^\backslash}, t) = \mathrm{E}\,(x_e - x_{ej}^{n} \oplus x_{ej}^{n} - x_{e^\backslash}, t)$

$= \mathrm{E}\,((x_{ej}^{n} - x_e) \oplus (x_{ej}^{n} - x_{e^\backslash}), t)$

$\geq \mathrm{E}\left(x_{ej}^{n} - x_e, \frac{t}{2}\right) * E(x_{ej}^{n} - x_{e^\backslash}, \frac{t}{2})$

$\geq (1 - \alpha) * (1 - \alpha)$

$= \min\,\{1 - \alpha, 1 - \alpha\}$

$= 1 - \alpha$

$\mathrm{E}\,(x_e - x_{e^\backslash}, t) \geq 1 - \alpha$

That implies $\lim_{n\to\infty} \mathrm{E}\,(x_e - x_{e^\backslash}, t) = 1$

$\mathrm{E}\,(x_e - x_{e^\backslash}, t) = 1$

By definition of fuzzy soft normed space

$\mathrm{E}\,(x_e - x_{e^\backslash}, t) = 1$ with $t > 0$ if and only if $x_e - x_{e^\backslash} = \theta_0$.

Hence $x_e = x_{e^\backslash}$

Definition(3.11): Let X and Y be two universe sets . We define two fuzzy soft (F , A) and (G,B) over universe sets , respectively . Let $f: X \to Y$ and $g: A \to B$ be two functions. Then the pair (f, g) is called Fuzzy soft function from (F,A) to (G,B) and denoted by (f, g): $(\mathrm{F}, \mathrm{A}) \to (G, B)$ if it is satisfies $f\big(F(x)\big) = G(g(x))$ for all $x \in A$.

The image of the fuzzy soft set (F,A) under the fuzzy soft function (f, g), denoted by (f, g)(F,A) =$(f(F), B)$, is a fuzzy soft set over universe set Y and defined by $\begin{cases} \vee g(x) = y\ f(F(x)) & if\ g^{-1}(y) \neq \emptyset \\ \emptyset & otherwise \end{cases}$

For all $x \in B$. The pre-image of fuzzy soft set (G,B) under fuzzy soft set (f, g) , denoted by $(f, g)^{-1}$(G,B) =$(f^{-1}(G), A)$ is fuzzy soft set over verse set X and defined by

Definition(3.12): Let (X , E,$\|.\|$) and (Y ,E,$\|.\|$) be two fuzzy soft normed spaces . A function f : $X \to Y$ is said to be fuzzy soft continuous at $x_0 \in X$ if for every sequence { X_{en}^n } in X with $\{X_{en}^n\} \to \{X_{e0}^0\}$ as $n \to \infty$ we have $f(X_{en}^n) \to f(X_{e0}^0)$ as $n \to \infty$. if f is fuzzy soft continuous at each vector of X then f is said to be fuzzy soft continuous function .

Theorem(3.13): If $(X, E, \|.\|)$ be a fuzzy soft normed space then

a) the function $(x_e, y_{e'}) \to x_e \oplus y_{e'}$ is continuous

b) the function (C, x_e) $\to$ $C * x_e$ is continuous where $x_{e1}, y_{e2} \in SSP(X)$ and C $\in$ K

Proof :

a) If $x_{en} \to x_e$ and $y_{e_n'} \to y_{e'}$ then as $n \to \infty$

$$E(x_{en} \oplus y_{e_n'}) - (x_e \oplus y_{e'}), t) = E(x_{en} \oplus y_{e_n'} - x_e - y_{e'}, t)$$

$$= E((x_{en} - x_e) \oplus (y_{e_n'} - y_{e'}), t)$$

$$= E(x_{en} - x_e, \tfrac{t}{2}) * E(y_{e_n'} - y_{e'}, t)$$

$$\to 1\ as\ t \to \infty$$

Thus the function(x_e , $y_{e'}$) $\to x_e \oplus y_{e'}$ is continuous

Definition(3.14): A fuzzy soft function f : $X \to Y$ is said to be fuzzy soft bounded ,if there exist a fuzzy soft real number M such that $\|f(x_e)\| \leq M \|x_e\|$ for all $x_e \in X$

Theorem(3.15) : The fuzzy soft function f : $X \to Y$ is fuzzy soft continuous if and only if it is fuzzy soft bounded .

Proof: Assume that f : $X \to Y$ be fuzzy soft continuous and f is not fuzzy soft bounded .Thus, there exist at least one sequence $\{X_{en}^n\}$ such that

$$\|f(X_{en}^n)\| \geq n \|X_{en}^n\| \qquad (1)$$

Where n is a fuzzy soft real number . It s clear that $x_{en}^{n} \neq \theta_0$.

Let us construct a fuzzy soft sequence as follow :

$$y_{en}^{n} = \frac{x_{en}^{n}}{n\|x_{en}^{n}\|}$$

It is clear that $y_{en}^{n} \to \theta_0$ as $n \to \infty$. Since f is fuzzy soft continuous , then we have $\|f(y_{en}^{n})\| \to 0 \; as\, n \to \infty$.

$$\|f(y_{en}^{n})\| = \left\|f \frac{x_{en}^{n}}{n\|x_{en}^{n}\|}\right\| = \frac{i}{n\|x_{en}^{n}\|} \|f(x_{en}^{n})\| > \frac{n \parallel X_{en}^{n} \parallel}{n \parallel X_{en}^{n} \parallel} = 1$$

Which is a contradiction

Conversely , suppose that f :$X \to Y$ is fuzzy soft bounded and the fuzzy soft sequence $\{X_{en}^{n}\}$ is convergent to the $\{X_{e0}^{0}\}$. In this case

$$\|f(X_{en}^{n}) - f(X_{e0}^{0})\| = \|f(X_{en}^{n} - X_{e0}^{0})\| \leq M\| X_{en}^{n} - X_{e0}^{0}\| \to 0$$

Which indicates that f is fuzzy soft continuous .

Definition(3.16): A fuzzy soft function f :$X \to Y$ is said to be fuzzy soft linear function if

1) f is additive , that is $f(x_e + y_e) = f(x_e) + f(y_e)$ for every $x_e, y_e \in X$

2) f is homogeneous ,that is , for every soft scalar r , $f(r\, x_e) = |r|\, f(x_e)$ for every $x_e \in X$,

Theorem(3.17) : Every fuzzy soft normed space is a fuzzy soft metric space.

Proof :

Define the fuzzy soft metric space by $\Delta(x_{e1}, y_{e2}, t) = E(x_{e_1} - y_{e_2}, t)$.......* for every $x_{e_1}, y_{e_2} \in X$.

Then it is clear to show that the fuzzy soft metric space axioms are satisfied .

1) $\Delta(x_{e1}, y_{e2}, t) = E(x_{e_1} - y_{e_2}, t) = 0$ if $t \leq 0$

2) $\Delta(x_{e1}, y_{e2}, t) = E(x_{e_1} - y_{e_2}, t) = 1 \; if \; t > 0$

3) $\Delta(x_{e1}, y_{e2}, t) = E(x_{e_1} - y_{e_2}, t)$

$=E(y_{e_2} - x_{e_1}, t)$

$=\Delta(y_{e_2}, x_{e_1}, t)$

$\Delta(x_{e1}, y_{e2}, t) = \Delta(y_{e_2}, x_{e_1}, t)$

4) $\Delta(x_{e1}, z_{e_3}, s \oplus t) = E(x_{e_1} - z_{e_3}, t \oplus s)$

$$= E(x_{e_1} - y_{e_2} + y_{e_2} - z_{e_3}, t \oplus s)$$

$$\geq E(x_{e_1} - y_{e_2}, s) * E(x_{e_1} - z_{e_3}, t)$$

$$= \Delta(x_{e_1}, y_{e_2}, s) * \Delta(x_{e_1}, z_{e_3}, t)$$

$\Delta(x_{e1}, z_{e_3}, s \oplus t) \geq \Delta(x_{e_1}, y_{e_2}, s) * \Delta(x_{e_1}, z_{e_3}, t)$

5) By the definition * of Δ we get Δ is continuous and $\Delta(x_{e_1}, y_{e_2}, .) : (0, \infty) \to [0,1]$

Theorem(3.18): Let $f: X \to Y$ be a fuzzy soft function .Then $\|f\|$ is fuzzy soft norm .

Theorem(3.19): Let $f: X \to Y$ and $g: Y \to Z$ be two fuzzy soft function .Then

a) $\|f \circ g\| \leq \|f\|\|g\|$;

b) If $f: X \to Y$ is fuzzy soft function then $\|f^n\| \leq \|f\|^n$.

Is satisfied .

Proof :

a) $\|f \circ g\| = \sup\{\|f \circ g(x_e)\| : \|x_e\| \leq 1\}$

$$= \sup\{\|f(g(x_e))\| : \|x_e\| \leq 1\}$$

$$\leq \sup\{\|f\|.\|g(x_e)\| : \|x_e\| \leq 1\}$$

$$\leq \|f\|\|g\|$$

b) If we take $f = g$ then we have $\|f^2\| \leq \|f\|^2$. Then $\|f^n\| \leq \|f\|^n$ is obtained.

References

[1] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy Soft Set ,Journal of Fuzzy Mathematics 9 (3) (2001) 589-602.

[2] B. Ahmad and Athar Kharal, On Fuzzy Soft sets, Advances in Fuzzy Systems, Volume 2009, Article ID 586507.

[3] Athar Kharal and B. Ahmad, Mappings on Fuzzy Soft Classes, Advances in Fuzzy Systems, Volume 2009, Article ID 407890. 3308-3314.

[4] B. Tanay, M. B. Kandemir, Topological structure of fuzzy soft sets, Computers and Mathematics with Applications 61 (2011) 2952-2957.

[5] J. Mahanta and P. K. Das, Results on fuzzy soft topological spaces, arXiv:1203.0634v1 [math.GM] 3 Mar 2012

[6] L.A.Zadeh, Fuzzy sets , Inf. Control, 8 (1965), 338-353