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Estimators of some inequality dynamical system

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Abstract

The aim of this paper to presented the estimation of some classes for some inequality dynamical system based on some inequality formulation that will make important role in solvability of these types of dynamical system.

1. Introduction

The inequalities played important role in the development of all branches of mathematics, the successfully of mathematical development availability of some kinds of inequalities and derivatives, differential equations have become a major tool in analysis of the differential equations that occur in nature and applications fields.

Most of the differential inequalities developed by many literature such as [1], [3],[5],[8] which is provide explicit known bounds on other equations and we have found wide sprouted types of inequalities have been mad a chive to develop various branches of mathematics such as differential equations to application in science and engineering practice also used the inequality to approximating various functions.

The applications of gronwall inequalities with nonlinear kernels of Lipschitz type to the problems of boundedness and convergence to zero at infinity of the solutions of certain volterra integral equations. The all types of stability are also investigated in[1],[7].

Our aim to make a new approach to compute the estimators of the solutions of inequality dynamical system with different methods that not explained before by using the approach of gronwall and generalization of gronwall inequalities in [1,3,5].

Lemma(1.1), [3]:

Let α, a, b and h be nonnegative constants and $u: [\alpha, \alpha+h] \to [0,\infty)$ be continuous if $0 \le u(t) \le \int_\alpha^t [bu(s)+a]ds$, $\alpha \le t \le \alpha+h$. Then $0 \le u(t) \le ahe^{bh}$, $\alpha \le t \le \alpha+h$.

Lemam(1.2), [3]:

Let α, β are any constants and c be nonnegative constants and $u, f: [\alpha, \beta] \to [0,\infty)$ be nonnegative continuous if $u(t) \le c + \int_\alpha^t f(s)u(s)ds$, $\alpha \le t \le \beta$.

Then $$u(t) \le c e^{\int_\alpha^t f(s)ds}, \ \alpha \le t \le \beta.$$

Lemma (1.3), [8]:

If $u(t)$ and $\alpha(t)$ are real valued continuous functions on $[a,b]$, $\alpha(t)$ is non-decreasing, and $\beta(t) \ge 0$ is integrable on $[a,b]$ with

$$u(t) \le \alpha(t) + \int_a^t \beta(s)u(s)ds, \ a \le t \le b.$$

Then $$u(t) \le \alpha(t)e^{\int_a^t \beta(s)ds}, \quad a \le t \le b.$$

Lemma (1.4), [6]:

Let u satisfies the inequality $u'(t) \le B(t)u(t)$, and bounded

Then $$u(t) \le u(a)e^{\int \ atB(s)ds}.$$

Lemma (1.5), [1]:

Let $w(s), u(s) \ge 0$ and $u(t) \le w(t) + \int_{t_0}^t v(s)u(s)ds.$

Then $$u(t) \le w(t) + \int_{t_0}^t v(s)w(s)e^{\int_s^t v(x)dx}ds.$$

Lemma (1.6), [4]:

If u satisfies the differential inequality

$u'(t) \ge f(u(t),t)$, and y a solution of $y'(t) = f(y(t),t)$ under the boundary $u(t_0) = y(t_0)$, for all $t < t_0$ $\quad u(t) \ge y(t)$.

Lemma(1.7),[7]:

Let $v(t)$ $\ t_0 \le t \le t_0 + a,$ piecewise continuous , u(t) and $u'(t)$ continuous on some interval . If $u'(t) \le v(t)u(t)$ then $u(t) \le u(t_0)e^{\int_{t_0}^t v(x)dx}$.

Remark(1.8),[1]

Let $u(t)\ \ a \le t \le b,$ be a real-valued function and suppose that $u'(t) \le Ku(t)$ where K is a constant then $u(t) \le u(a)e^{K(t-a)}, a \le t \le b.$

2. Main results:

The following results are presented in details to compute the estimators of the solutions of some types for dynamical systems with suitable spaces and conditions.

Theorem(2.1):

Let $x(.) \in C[a, a+h]$ and $x'(t) \le Ax(t) + b$, where $b \in R^{n\times 1}$, $A \in R^{n\times n}$ are semipositive matrices and $x(t) \in R^n$, a a constant and $0 \le x(t) \le x(a) + \int_a^t [Ax(s) + b]ds$. Then $0 \le x(t) \le hbe^{hA}$.

<u>Proof:</u>

Suppose that $x(t) = \left(z - z(a)\right)^T e^{A(t-a)}$ (1)

such that $\max z =$ maximum z at $t = t_1$, thus

$$0 \le \left(\max z^T - z(a)\right)e^{A(t_1-a)} \le \int_a^{t_1}\left[A\ \ \left(\max z(s) - z(a)\right)^T e^{A(s-a)} + b\right] ds$$

$$\le \left(\max z - z(a)\right)^T \int_a^{t_1} A\ \ e^{A(s-a)} ds + \int_a^{t_1} b ds$$

$$= \left[\left(\max z - z(a)\right)^T \left(e^{A(t_1-a)} - 1\right)\right] + b(t_1 - a).$$

Then

$\left(\max z - z(a)\right)e^{A(t_1-a)} \le \left(\max z - z(a)\right)e^{A(t_1-a)} - \left(\max z - z(a)\right) + b(t_1 - a)$, therefore

$\left(\max z - z(a)\right) \le b(t_1 - a) \le bh$, hence

From (1), we get

$x(t) = \left(\max z - z(a)\right)e^{Ah} \le bhe^{Ah}$.Therefore $x(t) \le bhe^{Ah}$.

Theorem (2.2):

Let $x(t) \in C\big[[a,b], R_0^+\big)$ and $x'(t) \leq A(t)x(t), \quad x(a) \geq 0$

where $A(t)$ is semipositive continuous matrices, then $x(t) \leq x(a)e^{\int_a^t A(s)\,ds}$, $\quad s \in J = [a,b]$.

Proof:

Define $z(a) = x(a)$, $\quad x(t) \leq z(t)$

and $\quad z'(t) = A(t)x(t) \leq A(t)z(t), \quad t \in J$ (2)

multiply (2) by $e^{-\int_a^t A(s)\,ds}$, we get

$$z'(t)e^{-\int_a^t A(s)\,ds} - A(t)z(t)e^{-\int_a^t A(s)\,ds} = \frac{d}{dt}\left[z(t)e^{\left(-\int_a^t A(s)\,ds\right)}\right] \leq 0$$

we obtain from (2)

$$\frac{d}{dt}\left[z(t)e^{\left(-\int_a^t A(s)\,ds\right)}\right] \leq 0 \tag{3}$$

Integrating of (3) from a to t , we get

$z(t)e^{-\int_a^t A(s)\,ds} - z(a) \leq 0$, since $(a) = x(a)$, we get

$z(t)e^{-\int_a^t A(s)\,ds} - x(a) \leq 0$, which implies that

$z(t)e^{-\int_a^t A(s)\,ds} \leq x(a)$, hence $\quad z(t) \leq x(a)e^{\int_a^t A(s)\,ds}$

So , since $x(t) \leq z(t) \leq x(a)e^{\int_a^t A(s)\,ds}$

Then $x(t) \leq x(a)e^{\int_a^t A(s)\,ds}$.

Theorem (*2.3*):

Let $x(t) \in C([a,b], R_0^+)$ and $f \; and \; G \in C([a,b], R_0^+)$

$$x'(t) \leq G(\tau)A(t)x(t), \quad \tau, t \in J = [a,b]$$

$$x(a) = f(t) \in C([a,b], R_0^+)$$

Then $\quad x(t) \leq f(t) + G(\tau)\int_a^t A(s)f(s)\, e^{\int_s^t A(\sigma)G(\sigma)d\sigma} ds.$

Proof;

Define a function $z(t)$ by $z(t) = \int_a^t A(s)x(s)ds$. Thus $\quad z(a) = 0$,

$x(t) \leq f(t) + G(\tau)\int_a^t A(s)x(s)ds$

Therefore $\quad x(t) \leq f(t) + G(\tau)z(t) \quad (4)$

So, $z'(t) = A(t)x(t) \leq A(t)f(t) + A(t)G(\tau)z(t) \quad (5)$

Multiply inequality equation (5) by $e^{-\int_a^t A(\sigma)G(\sigma)d\sigma}$, we have that

$$\frac{d}{dt}\left[z(t)e^{-\int_a^t A(\sigma)G(\sigma)d\sigma}\right] \leq A(t)f(t)e^{-\int_a^t A(\sigma)G(\sigma)d\sigma} \quad (6)$$

set $t = s$ in inequality equation (6), and integrated with respect to s from a to t , we obtain,

$$z(t)e^{-\int_a^t A(\sigma)G(\sigma)d\sigma} \leq \int_a^t A(s)f(s)e^{-\int_a^s A(\sigma)G(\sigma)d\sigma} ds$$

$z(t) \leq e^{\int_a^t A(\sigma)G(\sigma)d\sigma} \int_a^t A(s)f(s)e^{-\int_a^s A(\sigma)G(\sigma)d\sigma} ds$ =

$e^{\int_a^s A(\sigma)G(\sigma)d\sigma + e^{\int_s^t A(\sigma)G(\sigma)d\sigma}} \int_a^t A(s)f(s)e^{-\int_a^s A(\sigma)G(\sigma)d\sigma} ds$

from (4), we get $x(t) \leq f(t) + G(\tau)\int_a^t A(s)f(s)e^{\int_s^t A(\sigma)G(\sigma)d\sigma} ds$

Theorem (2.4):

Let $x(t) \in C([a,b], R)$ and $A(t-s)$ be a continuous semi positive matrix on $\Delta:\ a \leq s \leq t \leq b$.
$x'(t) \leq A(t-s)x(t), \quad x(a) = c$ Then $x(t) \leq c\, e^{\int_a^t A(t-s)ds}$.

Proof

Fix T such that $a \leq T \leq b$, then $a \leq t \leq T$, thus $x(t) \leq c + \int_a^t A(T-s)x(s)ds$

Now, let $z(t) = c + \int_a^t A(T-s)x(s)ds$, $z(a) = c$

$x(t) \leq z(t)$, for $a \leq t \leq T$

$z'(t) = A(T-s)x(t)$

$\leq A(T-s)z(t)$, for $a \leq t \leq T$ (7)

set $t = s$ in (1), and integrating with respect to s from a to t , we get

$$z(T) \leq c\, e^{\int_a^T A(T-s)\,ds} \quad (8)$$

since T is arbitrary then from (8) and $x(t) \leq z(t)$

$$x(T) \leq z(T) \leq c\, e^{\int_a^T A(T-s)\,ds}$$

Hence, $x(t) \leq c\, e^{\int_a^t A(t-s)\,ds}$. $a \leq t \leq T$

3. Conclusion

The estimators of some types of dynamical systems have been specified in some technical of inequality mathematical for integral and differential inequality that needed in studied the boundedness and stability of the trajectory of dynamical system.

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