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Generalized Gamma – Generalized Inverse Weibull Distribution

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Abstract: In this paper, we introduce a new family of continuous distributions based on generalized gamma. As a special case, generalized gamma - generalized inverse Weibull distribution is proposed. The probability density function, cumulative distribution function, reliability function and hazard rate function are introduced. Furthermore, most important statistical properties of generalized gamma - generalized inverse Weibull distribution such as the mean, variance, coefficient of skewness, coefficient of kurtosis, Shannon entropy, relative entropy and stress strength are obtained.

Keywords: Generalized Gamma Distribution, Generalized Inverse Weibull Distribution, Shannon Entropy, Relative Entropy, Stress-Strength Model.

1. Introduction

The generalized gamma distribution offers a useful stage, especially for the parametric analysis of survival and reliability data, that lies in three main reasons (1) its available in most standard statistical packages (2) has several particular cases, such as the exponential, gamma, Weibull, log-normal, Maxwell-Boltzmann and Rayleigh distributions (3) its hazard functions include different basic shapes (increasing, decreasing, bathtub and unimodal or arc-shaped)[1]. For more details about the generalized gamma distribution see [3].

However, over the last two decades, according to extending common classes (families) of continuous distributions, many generalized classes of distributions have been proposed and studied for modeling data in different applied areas such as, engineering, physics, medical sciences, economics, finance, biological and environmental studies. These generalized distributions seek to give more and more flexibility by adding one (or more) parameters to the baseline distribution. The generalized class were pioneered to Marshall and Olkin (1997)[5] who proposed the Marshall-Olkin-G and to Gupta et al. (1998)[4] who proposed the exponentiated-G class. Many other generalized classes cited via Yousof et al. (2017)[8].

In this paper, a new family of distributions based on generalized gamma, named generalized gamma$-$ G distributions, has been proposed. The probability density function and cumulative distribution function of this new family are introduced in Section 2. The generalized gamma-generalized inverse Weibull as a special case along with some its important properties are presented in Sections 3 and 4. The Shannon entropy, relative entropy and stress - strength of generalized gamma-generalized inverse Weibull random variable are discussed in Sections 5, 6 and 7 respectively. Finally, in Section 8, some conclusions are addressed.

2. Generalized Gamma – G Distributions

The generalized gamma (GG) distribution was introduced by Stacy [7]. The cumulative distribution function (cdf) of the GG distribution is given by [6],

$$H(x;a,d,p)=\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right]}{\Gamma\left(\frac{d}{p}\right)}\ ;\ x>0\ \text{ and }\ a,d,p>0 \qquad (1)$$

and the corresponding probability distribution function (pdf) is given by [6],

$$h(x;a,d,p) = \frac{1}{\Gamma\left(\frac{d}{p}\right)} \left(\frac{p}{a^d}\right) x^{d-1} \, exp\left[-\left(\frac{x}{a}\right)^p\right] \;\; ; \quad x > 0 \text{ and } a,d,p > 0 \qquad (2)$$

where $\gamma(\cdot\,,\,\cdot)$ is the incomplete gamma function, $\Gamma(\cdot)$ is the gamma function, , a is the scale parameter, d and p are the shape parameters.

Now, suppose that $G(x)$ and $g(x)$ are the baseline cdf and pdf of a random variable X. The proposed new class of distributions is given by,

$$1 - F(x) = R(x) = \int_0^{-\ln G(x)} h(x)\, dx = H(-\ln G(x)) \qquad (3)$$

So, the proposed cdf for this new class of distributions will be,

$$F(x) = 1 - H(-\ln G(x)) \qquad (4)$$

and the associated pdf, $f(x) = \frac{d}{dx}[F(x)] = -\frac{d}{dx}[R(x)]$, for the new class of distributions will be,

$$f(x) = \frac{g(x)}{G(x)}\, h(-\ln G(x)) \qquad (5)$$

Now, from (4) and (5) above, a new family of continuous distributions based on interval $[0,\infty)$ generalized gamma distribution, named generalized gamma$-$ G distributions, have been proposed, where,

$$H(-\ln G(x)) = \frac{\gamma\left[\frac{d}{p},\left(\frac{-\ln G(x)}{a}\right)^p\right]}{\Gamma\left(\frac{d}{p}\right)} \qquad (6)$$

$$h(-\ln G(x)) = \frac{1}{\Gamma\left(\frac{d}{p}\right)} \left(\frac{p}{a^d}\right) (-\ln G(x))^{d-1} \exp\left[-\left(\frac{-\ln G(x)}{a}\right)^p\right] \qquad (7)$$

3. Generalized Gamma-Generalized Inverse Weibull Distribution

de Gusmão et al.[2] introduced a new lifetime distribution with three parameter, named the generalized inverse Weibull (GIW) distribution. The GIW distribution is much more flexible than the original inverse Weibull distribution and could have decreasing, increasing and unimodal hazard rate functions.

Suppose that $G(x)$ and $g(x)$ represents the cdf and pdf of GIW distribution given by [2],

$$G(x) = \exp\left[-\theta\left(\frac{\alpha}{x}\right)^\beta\right] \qquad (8)$$

$$g(x) = \theta\beta\alpha^\beta\, x^{-(\beta+1)} \exp\left[-\theta\left(\frac{\alpha}{x}\right)^\beta\right] \qquad (9)$$

Depending on (6) and (7), $H(-\ln G(x))$ and $h(-\ln G(x))$, will be,

$$H(-\ln G(x)) = \frac{\gamma\left[\frac{d}{p},\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^\beta\right)^p\right]}{\Gamma\left(\frac{d}{p}\right)}$$

$$h(-\ln G(x)) = \frac{1}{\Gamma\left(\frac{d}{p}\right)}\left(\frac{p}{a^d}\right)\left[\theta\left(\frac{\alpha}{x}\right)^{\beta}\right]^{d-1} exp\left[-\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p}\right]$$

Then, the cdf and pdf in (6) and (7) of a new proposed distribution named generalized gamma-generalized inverse Weibull ($GG-GIW$) distribution are given by,

$$F(x) = 1 - \frac{\gamma\left[\frac{d}{p},\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p}\right]}{\Gamma\left(\frac{d}{p}\right)} \quad (10)$$

$$f(x) = \theta^d\,\beta\,\alpha^{\beta d}\frac{\left(\frac{p}{a^d}\right)}{\Gamma\left(\frac{d}{p}\right)}\; x^{-(1+\beta d)}\; exp\left[-\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p}\right] \quad (11)$$

The reliability and hazard rate functions of $GG-GIW$ distribution are given respectively by,

$$R(x) = 1 - F(x) = \frac{\gamma\left[\frac{d}{p},\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p}\right]}{\Gamma\left(\frac{d}{p}\right)} \quad (12)$$

$$A(x) = \frac{f(x)}{R(x)} = \frac{\theta^d\,\beta\,\alpha^{\beta d}\left(\frac{p}{a^d}\right)x^{-(1+\beta d)}\; exp\left[-\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p}\right]}{\gamma\left[\frac{d}{p},\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p}\right]} \quad (13)$$

4. Properties of the GG-GIW Distribution

The r^{th} moment, $E(X^r) = \int_0^\infty x^r f(x)\,dx$, of $GG-GIW$ distribution is,

$$E(X^r) = \int_0^\infty x^r\,\theta^d\beta\,\alpha^{\beta d}\frac{\left(\frac{p}{a^d}\right)}{\Gamma\left(\frac{d}{p}\right)}\; x^{-(1+\beta d)}\; exp\left[-\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p}\right]dx$$

$$E(X^r) = \theta^d\,\beta\,\alpha^{\beta d}\frac{\left(\frac{p}{a^d}\right)}{\Gamma\left(\frac{d}{p}\right)}\int_0^\infty x^{r-(1+\beta d)}\; exp\left[-\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p}\right]dx$$

Let $u = \left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p} = \left(\frac{\theta}{a}\right)^{p}\left(\frac{\alpha}{x}\right)^{\beta p} \rightarrow\; u(0) = \infty,\; u(\infty) = 0,$

$du = -\left(\frac{\theta}{a}\right)^{p}\beta\,p\,\alpha^{\beta p}x^{-(\beta p+1)}dx$

Also that $u = \left(\frac{\theta}{a}\right)^{p}\left(\frac{\alpha}{x}\right)^{\beta p} \rightarrow \sqrt[\beta p]{\frac{u}{\left(\frac{\theta}{a}\right)^{p}}} = \frac{\alpha}{x} \rightarrow\; x = \frac{\alpha}{\sqrt[\beta p]{\frac{u}{\left(\frac{\theta}{a}\right)^{p}}}} = \frac{\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}}{u^{\frac{1}{\beta p}}}$

So that, $du = -\left(\frac{\theta}{a}\right)^{p} \beta\, p\, \alpha^{\beta p} \left[\frac{\alpha \left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}}{u^{\frac{1}{\beta p}}}\right]^{-(\beta p+1)} dx$

$$du = -\left(\frac{\theta}{a}\right)^{p} \beta\, p\, \alpha^{\beta p} \left[\frac{u^{\frac{1}{\beta p}}}{\alpha \left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}}\right]^{(\beta p+1)} dx$$

$$dx = \frac{-\left(\frac{\alpha \left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}}{u^{\frac{1}{\beta p}}}\right)^{\beta p+1}}{\left(\frac{\theta}{a}\right)^{p} \beta\, p\, \alpha^{\beta p}}\, du = -\frac{\alpha}{\beta p} \left(\frac{\theta}{a}\right)^{\frac{1}{\beta}} u^{-\left(1+\frac{1}{\beta p}\right)} du$$

Then,

$$E(X^r) = \theta^d \beta\, \alpha^{\beta d} \frac{\left(\frac{p}{a^d}\right)}{\Gamma\left(\frac{d}{p}\right)} \int_0^\infty \left[\frac{\alpha \left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}}{u^{\frac{1}{\beta p}}}\right]^{r-(1+\beta d)} exp[-u]\left(\frac{\alpha}{\beta p}\right)\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}} u^{-\left(1+\frac{1}{\beta p}\right)} du$$

$$E(X^r) = \frac{\left(\frac{\theta}{a}\right)^{\frac{r}{\beta}} \alpha^r}{\Gamma\left(\frac{d}{p}\right)} \int_0^\infty u^{\left(1-\frac{r}{\beta p}\right)-1} e^{-u} du$$

Therefor the r^{th} moment is given by:

$$E(X^r) = \frac{\left(\frac{\theta}{a}\right)^{\frac{r}{\beta}} \alpha^r}{\Gamma\left(\frac{d}{p}\right)} \Gamma\left[1-\frac{r}{\beta p}\right] \qquad (14)$$

In particularly,

$$E(X) = \frac{\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}} \alpha}{\Gamma\left(\frac{d}{p}\right)} \Gamma\left[1-\frac{1}{\beta p}\right] \qquad (15)$$

$$E(X^2) = \frac{\left(\frac{\theta}{a}\right)^{\frac{2}{\beta}} \alpha^2}{\Gamma\left(\frac{d}{p}\right)} \Gamma\left[1-\frac{2}{\beta p}\right] \qquad (16)$$

$$E(X^3) = \frac{\left(\frac{\theta}{a}\right)^{\frac{3}{\beta}} \alpha^3}{\Gamma\left(\frac{d}{p}\right)} \Gamma\left[1-\frac{3}{\beta p}\right] \qquad (17)$$

$$E(X^4) = \frac{\left(\frac{\theta}{a}\right)^{\frac{4}{\beta}} \alpha^4}{\Gamma\left(\frac{d}{p}\right)} \Gamma\left[1 - \frac{4}{\beta p}\right]$$
(18)

So that, the mean and variance of $GG - GIW$ random variable can be obtain as,

$$mean = \mu = E(X) = \frac{\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}} \alpha}{\Gamma\left(\frac{d}{p}\right)} \Gamma\left[1 - \frac{1}{\beta p}\right] \quad (19)$$

$$Var(x) = \sigma^2 = E(X^2) - \mu^2 = \frac{\left(\frac{\theta}{a}\right)^{\frac{2}{\beta}} \alpha^2}{\Gamma\left(\frac{d}{p}\right)} \Gamma\left[1 - \frac{2}{\beta p}\right] - \left[\frac{\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}} \alpha}{\Gamma\left(\frac{d}{p}\right)} \Gamma\left[1 - \frac{1}{\beta p}\right]\right]^2 \quad (20)$$

The coefficients of skewness, sk and kurtosis, kr of $GG - GIW$ random variable can be obtain as,

$$sk = \frac{E(X-\mu)^3}{\sigma^3} = \frac{E(X^3) - 3\mu\, E(X^2) + 2\mu^3}{[\sigma^2]^{\frac{3}{2}}} \quad (21)$$

$$kr = \frac{E(X-\mu)^4}{(\sigma^2)^2} = \frac{E(X^4) - 4\mu\, E(X^3) + 6\mu^2 E(X^2) - 3\mu^3}{(\sigma^2)^2} - 3 \quad (22)$$

where μ , σ^2 as in $(19), (20)$and $E(X^2), E(X^3), E(X^4)$ as in $(16), (17)$and (18)reseptivelly.

The characteristic function $Q_X(t) = E(e^{itx}) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(X^r)$ of $GG - GIW$ random variable can be obtain as,

$$Q_X(t) = \sum_{r=0}^{\infty} \frac{(it\alpha)^r \left(\frac{\theta}{a}\right)^{\frac{r}{\beta}}}{r!\,\Gamma\left(\frac{d}{p}\right)} \Gamma\left[1 - \frac{r}{\beta p}\right] \quad (23)$$

5. Shannon Entropy of $GG - GIW$ Random Variable

The formula of Shannon entropy can be written as [8],

$$H(x) = -\int_0^{\infty} f(x) \ln f(x)\; dx \quad (24)$$

Since,

$$\ln f(x) = \ln\left\{\theta^d\, \beta\, \alpha^{\beta d} \frac{\left(\frac{p}{a^d}\right)}{\Gamma\left(\frac{d}{p}\right)}\; x^{-(1+\beta d)}\; exp\left[-\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p}\right]\right\}$$

$$\ln f(x) = \ln\left\{\theta^d \beta\, \alpha^{\beta d} \frac{\left(\frac{p}{a^d}\right)}{\Gamma\left(\frac{d}{p}\right)}\right\} - (1 + \beta d) \ln x - \left(\frac{\theta}{a}\right)^{p} \alpha^{\beta p} x^{-\beta p}$$

So that, Eq. (24) will be,

$$H(x) = -ln\left\{\theta^d \beta\, \alpha^{\beta d} \frac{\left(\frac{p}{a^d}\right)}{\Gamma\left(\frac{d}{p}\right)}\right\} + (1+\beta d)\int_0^\infty ln\, x\, f(x)\, dx \;+ \left(\frac{\theta}{a}\right)^p \alpha^{\beta p} E\left(X^{-\beta p}\right) \qquad (25)$$

Let $I = \int_0^\infty ln\, x\, f(x)\, dx$, so that,

$$I = \int_0^\infty ln\, x\;\; \theta^d\, \beta\, \alpha^{\beta d} \frac{\left(\frac{p}{a^d}\right)}{\Gamma\left(\frac{d}{p}\right)}\;\; x^{-(1+\beta d)}\;\, exp\left[-\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^\beta\right)^p\right] dx$$

Using the transformation, $u = \left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^\beta\right)^p$, as in section 4, then,

$$I = \int_0^\infty \ln\left(\frac{\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}}{u^{\frac{1}{\beta p}}}\right)\; \theta^d\, \beta\, \alpha^{\beta d} \frac{\left(\frac{p}{a^d}\right)}{\Gamma\left(\frac{d}{p}\right)} \left(\frac{\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}}{u^{\frac{1}{\beta p}}}\right)^{-(1+\beta d)} exp[-u]\frac{\alpha}{\beta p}\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}} u^{-\left(1+\frac{1}{\beta p}\right)} du$$

$$I = \frac{1}{\Gamma\left(\frac{d}{p}\right)}\int_0^\infty \left[ln\left(\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}\right) - \frac{1}{\beta p} ln\, u\right] u^{\frac{d}{p}-1}\; e^{-u}\, du$$

$$I = \frac{1}{\Gamma\left(\frac{d}{p}\right)}\left[ln\left(\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}\right)\int_0^\infty u^{\frac{d}{p}-1}\, e^{-u}\, du \;\; - \frac{1}{\beta p}\int_0^\infty ln\, u\;\, u^{\frac{d}{p}-1}\, e^{-u}\, du\right]$$

By using, $\int_0^\infty x^{s-1}\, ln\, x\;\; e^{-mx}\, dx = m^{-s}\, \Gamma(s)(\psi(s) - ln\, m)$, we get,

$$I = \frac{1}{\Gamma\left(\frac{d}{p}\right)}\left[ln\left(\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}\right)\Gamma\left(\frac{d}{p}\right) - \frac{1}{\beta p}\Gamma\left(\frac{d}{p}\right)\left(\psi\left(\frac{d}{p}\right) - ln\, 1\right)\right]$$

$$I = ln\left(\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}\right) - \frac{1}{\beta p}\psi\left(\frac{d}{p}\right) \qquad (26)$$

and,

$$E\left[X^{-\beta p}\right] = \int_0^\infty x^{-\beta p}\, \theta^d\, \beta\, \alpha^{\beta d} \frac{\left(\frac{p}{a^d}\right)}{\Gamma\left(\frac{d}{p}\right)}\;\; x^{-(1+\beta d)}\;\, exp\left[-\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^\beta\right)^p\right] dx$$

$$E\left[X^{-\beta p}\right] = \frac{1}{\Gamma\left(\frac{d}{p}\right)}\int_0^\infty \left(\frac{\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}}{u^{\frac{1}{\beta p}}}\right)^{-\beta p} \left(\frac{1}{u^{\frac{1}{\beta p}}}\right)^{-(1+\beta d)} exp[-u]\, u^{-\left(1+\frac{1}{\beta p}\right)} du$$

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$$E\left[X^{-\beta p}\right] = \frac{\Gamma\left[\frac{d}{p}+1\right]}{\Gamma\left(\frac{d}{p}\right)} \frac{1}{\alpha^{\beta p}\left(\frac{\theta}{a}\right)^{p}} = \frac{\frac{d}{p}}{\alpha^{\beta p}\left(\frac{\theta}{a}\right)^{p}} \quad (27)$$

Now substituting (26) and (27)in (25), $H(x)$ will be,

$$H(x) = -\ln\left\{\theta^{d}\beta\alpha^{\beta d}\frac{\left(\frac{p}{a^{d}}\right)}{\Gamma\left(\frac{d}{p}\right)}\right\} + (1+\beta d)\left[\ln\left(\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}\right) - \frac{1}{\beta p}\psi\left(\frac{d}{p}\right)\right] + \left(\frac{\theta}{a}\right)^{p}\alpha^{\beta p}\frac{\frac{d}{p}}{\alpha^{\beta p}\left(\frac{\theta}{a}\right)^{p}}$$

Therefor the Shannon entropy of $GG-GIW$ random variable can be written as,

$$H(x) = \frac{d}{p} - \ln\left\{\theta^{d}\beta\alpha^{\beta d}\frac{\left(\frac{p}{a^{d}}\right)}{\Gamma\left(\frac{d}{p}\right)}\right\} + (1+\beta d)\left[\ln\left(\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}\right) - \frac{1}{\beta p}\psi\left(\frac{d}{p}\right)\right] \quad (28)$$

6. Relative Entropy of $GG-GIW$ Random Variable

The relative entropy can be written as,

$$Dkl(f\|f_1) = \int_0^{\infty} f(x)\ln\frac{f(x)}{f_1(x)}\,dx \quad (29)$$

Since,

$$\ln\frac{f(x)}{f_1(x)} = \ln\left[\frac{\theta^{d}\,\beta\alpha^{\beta d}\frac{\left(\frac{p}{a^{d}}\right)}{\Gamma\left(\frac{d}{p}\right)}\;x^{-(1+\beta d)}\;\exp\left[-\left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p}\right]}{{\theta_1}^{d_1}\,\beta_1{\alpha_1}^{\beta_1 d_1}\frac{\left(\frac{p_1}{{a_1}^{d_1}}\right)}{\Gamma\left(\frac{d_1}{p_1}\right)}\;x^{-(1+\beta_1 d_1)}\;\exp\left[-\left(\frac{\theta_1}{a_1}\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)^{p_1}\right]}\right]$$

Let $w = \dfrac{\theta^{d}\,\beta\alpha^{\beta d}\frac{\left(\frac{p}{a^{d}}\right)}{\Gamma\left(\frac{d}{p}\right)}}{{\theta_1}^{d_1}\,\beta_1{\alpha_1}^{\beta_1 d_1}\frac{\left(\frac{p_1}{{a_1}^{d_1}}\right)}{\Gamma\left(\frac{d_1}{p_1}\right)}}$, then,

$$\ln\frac{f(x)}{f_1(x)} = \ln w - (1+\beta d)\ln x + (1+\beta_1 d_1)\ln x - \left(\frac{\theta}{a}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{p} + \left(\frac{\theta_1}{a_1}\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)^{p_1}$$

$$\ln\frac{f(x)}{f_1(x)} = \ln w + (\beta_1 d_1 - \beta d)\ln x - \left(\frac{\theta}{a}\right)^{p}\alpha^{\beta p}x^{-\beta p} + \left(\frac{\theta_1}{a_1}\right)^{p_1}\alpha_1^{\beta_1 p_1}x^{-\beta_1 p_1}$$

Now,

$$Dkl(f\|f_1) = \ln w + (\beta_1 d_1 - \beta d)\int_0^{\infty}\ln x\, f(x)\,dx - \left(\frac{\theta}{a}\right)^{p}\alpha^{\beta p}E\left[X^{-\beta p}\right] + \left(\frac{\theta_1}{a_1}\right)^{p_1}\alpha_1^{\beta_1 p_1}E\left[X^{-\beta_1 p_1}\right] \quad (30)$$

From (26) and (27), we get respectively that, $\int_0^\infty \ln x\, f(x)\, dx = \ln\left(\alpha \left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}\right) - \frac{1}{\beta p}\psi\left(\frac{d}{p}\right)$ and

$E\left[X^{-\beta p}\right] = \frac{\Gamma\left[\frac{d}{p}+1\right]}{\Gamma\left(\frac{d}{p}\right)} \frac{1}{\alpha^{\beta p}\left(\frac{\theta}{a}\right)^{p}} = \frac{\frac{d}{p}}{\alpha^{\beta p}\left(\frac{\theta}{a}\right)^{p}}$. Also, we can get that,

$$E\left[X^{-\beta_1 p_1}\right] = \frac{\Gamma\left[\frac{d}{p}+\frac{\beta_1 p_1}{\beta p}\right]}{\Gamma\left(\frac{d}{p}\right)} \frac{1}{\alpha^{\beta_1 p_1}\left(\frac{\theta}{a}\right)^{\frac{\beta_1 p_1}{\beta}}} \qquad (31)$$

Now substituting (26), (27) and (31) in (30), $Dkl(f\|f_1)$ will be,

$$Dkl(f\|f_1) = \ln w + (\beta_1 d_1 - \beta d)\left[\ln\left(\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}\right) - \frac{1}{\beta p}\psi\left(\frac{d}{p}\right)\right] - \left(\frac{\theta}{a}\right)^{p} \alpha^{\beta p} \frac{\frac{d}{p}}{\alpha^{\beta p}\left(\frac{\theta}{a}\right)^{p}} + \left(\frac{\theta_1}{a_1}\right)^{p_1} \alpha_1^{\beta_1 p_1} \frac{\Gamma\left[\frac{d}{p}+\frac{\beta_1 p_1}{\beta p}\right]}{\Gamma\left(\frac{d}{p}\right)} \frac{1}{\alpha^{\beta_1 p_1}\left(\frac{\theta}{a}\right)^{\frac{\beta_1 p_1}{\beta}}}$$

Therefor the relative entropy of $GG - GIW$ random variable can be written as,

$$Dkl(f\|f_1) = \ln\left[\frac{\theta^{d}\, \beta\, \alpha^{\beta d} \frac{\left(\frac{p}{a^{d}}\right)}{\Gamma\left(\frac{d}{p}\right)}}{{\theta_1}^{d_1}\, \beta_1 \alpha_1^{\beta_1 d_1} \frac{\left(\frac{p_1}{a_1^{d_1}}\right)}{\Gamma\left(\frac{d_1}{p_1}\right)}}\right] + (\beta_1 d_1 - \beta d)\left[\ln\left(\alpha\left(\frac{\theta}{a}\right)^{\frac{1}{\beta}}\right) - \frac{1}{\beta p}\psi\left(\frac{d}{p}\right)\right] - \frac{d}{p} + \frac{\Gamma\left[\frac{d}{p}+\frac{\beta_1 p_1}{\beta p}\right]}{\Gamma\left(\frac{d}{p}\right)} \frac{\left(\frac{\theta_1}{a_1}\right)^{p_1} \alpha_1^{\beta_1 p_1}}{\alpha^{\beta_1 p_1}\left(\frac{\theta}{a}\right)^{\frac{\beta_1 p_1}{\beta}}} \qquad (32)$$

7. Stress - Strength Model

Stress-strength model is the most generally approach employed for dependability estimation. This model is used in various applications especially with engineering such as strength failure and system collapse[8].

Let Y and X be the stress strength random variable independent of each other follow respectively $GG - GIW$ with two different parameter.

$$R = P(Y < X) = \int_0^\infty f_X(x)\ F_Y(x)dx \qquad (33)$$

$$R = \int_0^\infty f_X(x)\left[1 - \frac{\gamma\left[\frac{d_1}{p_1}, \left(\frac{\theta_1}{a_1}\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)^{p_1}\right]}{\Gamma\left(\frac{d_1}{p_1}\right)}\right]dx$$

$$R = \int_0^\infty f_X(x)\, dx - \frac{1}{\Gamma\left(\frac{d_1}{p_1}\right)}\int_0^\infty \gamma\left[\frac{d_1}{p_1}, \left(\frac{\theta_1}{a_1}\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)^{p_1}\right] f_X(x)\, dx$$

$$R = 1 - \frac{1}{\Gamma\left(\frac{d_1}{p_1}\right)} \int_0^\infty \gamma\left[\frac{d_1}{p_1}, \left(\frac{\theta_1}{a_1}\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)^{p_1}\right] f_X(x)\, dx \quad (34)$$

Since, $\gamma[s,u] = \sum_{k=0}^{\infty} \frac{(-1)^k u^{s+k}}{k!(s+k)} = u^s \sum_{k=0}^{\infty} \frac{(-u)^k}{k!(s+k)}$, so that,

$$\gamma\left[\frac{d_1}{p_1}, \left(\frac{\theta_1}{a_1}\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)^{p_1}\right] = \left[\left(\frac{\theta_1}{a_1}\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)^{p_1}\right]^{\frac{d_1}{p_1}} \sum_{k=0}^{\infty} \frac{\left[-\left(\frac{\theta_1}{a_1}\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)^{p_1}\right]^k}{k!\left(\frac{d_1}{p_1}+k\right)}$$

$$\gamma\left[\frac{d_1}{p_1}, \left(\frac{\theta_1}{a_1}\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)^{p_1}\right] = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\theta_1}{a_1}\right)^{kp_1+d_1} \alpha_1^{\beta_1(kp_1+d_1)}}{k!\left(\frac{d_1}{p_1}+k\right)} \; x^{-\beta_1(kp_1+d_1)} \quad (35)$$

Now substituting (35)in (34), R will be,

$$R = 1 - \frac{1}{\Gamma\left(\frac{d_1}{p_1}\right)} \int_0^\infty \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\theta_1}{a_1}\right)^{kp_1+d_1} \alpha_1^{\beta_1(kp_1+d_1)}}{k!\left(\frac{d_1}{p_1}+k\right)} \; x^{-\beta_1(kp_1+d_1)} \; f_X(x)\, dx$$

$$R = 1 - \frac{1}{\Gamma\left(\frac{d_1}{p_1}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\theta_1}{a_1}\right)^{kp_1+d_1} \alpha_1^{\beta_1(kp_1+d_1)}}{k!\left(\frac{d_1}{p_1}+k\right)} E\left[X^{-\beta_1(kp_1+d_1)}\right]$$

where,

$$E\left[X^{-\beta_1(kp_1+d_1)}\right] = \frac{\Gamma\left[\frac{d}{p} + \frac{\beta_1(kp_1+d_1)}{\beta p}\right]}{\Gamma\left(\frac{d}{p}\right)} \frac{1}{\alpha^{\beta_1(kp_1+d_1)} \left(\frac{\theta}{a}\right)^{\frac{\beta_1(kp_1+d_1)}{\beta}}} \quad (36)$$

Therefor the stress - strength of $GG - GIW$ random variable can be written as,

$$R = 1 - \frac{1}{\Gamma\left(\frac{d}{p}\right)\Gamma\left(\frac{d_1}{p_1}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\theta_1}{a_1}\right)^{kp_1+d_1} \alpha_1^{\beta_1(kp_1+d_1)}}{k!\left(\frac{d_1}{p_1}+k\right) \alpha^{\beta_1(kp_1+d_1)} \left(\frac{\theta}{a}\right)^{\frac{\beta_1(kp_1+d_1)}{\beta}}} \Gamma\left[\frac{d}{p} + \frac{\beta_1(kp_1+d_1)}{\beta p}\right] \quad (37)$$

8. Concluding Remarks

In this paper, we presented a new family of continuous distributions based on generalized gamma. The generalized gamma-generalized inverse Weibull (GG-GIW) distribution is discussed as special case of this new family. The most important properties of GG-GIW is derived. We provide form for characteristic function, rth moment, mean, variance, skewness, kurtosis, reliability function, hazard rate function, Shannon entropy along with relative entropy. This paper deals also with the determination of stress-strength $R = P(y < x)$ when X (strength) and Y (stress) are two independent GG-GIW distribution with different parameters.

المستخلص: في هذا البحث سنقوم بتقديم عائلة جديدة من التوزيعات المستمرة بالاستناد على توزيع كاما المعمم . كحالة خاصة ، تم اقتراح توزيع كاما المعمم- معكوس ويبل المعمم. تم تقديم دالة الكثافة الاحتمالية ، دالة التوزيع التراكمية ، دالة المعولية ودالة المخاطرة . فضلا عن ذلك ، تم تقديم اغلب الخصائص الاحصائية المهمة للتوزيع المقترح كـ المتوسط ، التباين ، معامل الالتواء ، معامل التفلطح كذلك دالة انتروبي شانون والانتروبي النسبية والاجهاد/ المرونة .

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