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# Compact Linear Operator on Modular Spaces

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**Abstract:**

This study tackles Compact linear operator on modular space , we have introduced the definition of Compact linear operator on modular space and also define bounded and continuous linear operator on modular space and proves some new results related with them .

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**Keywords**: modular space , Compact linear operator on modular space , bounded and continuous linear operator on modular space .

## 1.Introduction

The theory of modular space was introduction by Nakano [2] in 1950 in the connection with the theory of order spaces and redefined and generalization by Musielak and Orlicz [4 ] in 1959 , many other mathematicians have studied modular space from several point of view , there are is a large set of known application of modular space in various part of analysis , probability and mathematical statistics .

In this effort , we introduction the notion Compact linear operator on modular space and bounded linear operator on modular space and also define continuous linear operator in modular space .

## 2. Main Results

**Definition (2. 1)[5]:**

Let $X$ be a linear space over afield $\mathbb{F}$. A function $M: X \to [0,\infty]$ is called modular if:

1. $M(x) = 0 \Leftrightarrow x = 0$ .

2. $M(\alpha x) = M(x)$ for $\alpha \in F$ with $|\alpha| = 1$ , for all $\alpha \in F$.

3. $M(\alpha x + \beta y) \leq M(x) + M(y)$ iff $\alpha , \beta \geq 0$ ,for all $x \in X$.

**Example (2.2 ) [1]:**

Let $X = R^2$ with $(x, y) = |x| + |y|$ , for any pair $(x - y)$ in $X$, then $X_M$ is modular space .

**Example (2.3 ) [5]:**

As a classical example we mention to the Orlicz modular defined for every measurable real function $f$ by the formula

$$M(f) = \int \emptyset(|f(t)|)\, d\mu(t),$$

where $\mu$ denotes the Lebesgue`s measure in $\mathbb{R}$ and $\emptyset: \mathbb{R} \rightarrow [0,\infty)$ is continuous . we also assume that $\emptyset(y) = 0$ if and only if $y = 0$ and $\emptyset(t) \rightarrow \infty$ as $t \rightarrow \infty$

**Definition (2.4) :**

Let $X$ and $Y$ be a modular spaces and $T: X \rightarrow Y$ a linear operator .The operator $T$ is said to be bounded if there exist a real number $r$ Such that $M\big(T(x)\big) \leq rM(x)$.

**Example (2.5) :**

1. The identity operator $I: X \rightarrow X$ on a modular space $X \neq \{0\}$ is bounded.
2. The zero operator $0: X \rightarrow Y$ on a modular space $X$ is bounded .

**Definition (2.6) :**

Let $X$ and $Y$ be a modular spaces. A operator $T : X \rightarrow Y$ is called compact linear operator if for every bounded subset $B$ of $X$ that $\overline{T(B)}$ is compact in $Y$.

**Definition (2.7) :**

Let $(X, M_1)$ and $(Y, M_2)$ be a modular spaces over the same field $F$, The operator $T: (X, M_1) \rightarrow (Y, M_2)$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon \in (0,1)$ and all $t > 0$ there exist $\delta \in (0,1)$ such that for all $x \in X$:

$$M_1(x - x_0) < \delta \implies M_2\big(T(x) - T(x_0)\big) < \varepsilon$$

**Lemma (2.8) :**

Let $X$ and $Y$ be a modular spaces , then Every compact linear operator $T: X \rightarrow Y$ is bounded , hence continuous .

**Theorem (2.9) :**

Let $T: X \rightarrow Y$ be a linear operator and $X, Y$ are modular spaces .Then

1. $T$ is continuous if and only if $T$ is bounded .
2. If $T$ is continuous at a single point ,it is bounded .

Journal of Iraqi Al-Khwarizmi Society (JIKhS) Volume:3 Issue:Special 2019 pages: 1-7

**Definition (2.10) [3]:**

Let $T: X \longrightarrow Y$ be a linear operator ,then null space of $T$ is the set of all $x \in X$ such that $Tx = 0$ .

**Corollary (2.11) [3]:**

Let $T$ be a bounded linear operator .Then :

1. $x_n \to x$ implies $T_{x_n} \to T_x$.
2. The null spaces $\mathcal{N}(T)$ is closed .

**Definition (2.12) :**

Let $(X, M)$ be a modular space. A subset $A$ of $X$ is said to be compact if any sequence $\{x_n\}$ in $A$ has a subsequence converging to an element of $A$.

**Theorem (1.13 ) :**

Let $(X, M)$ be a modular spaces ,then

1 . If $x_n \to x$ , $y_n \to y$ ,then $x_n + y_n \to x + y$.

2 . If $x_n \to x$ then $x_n \to cx, c \in F/\{0\}$ .

**Proof :**

1. Let $x_n \to x$ and $y_n \to y$

$$M((x_n + y_n) - (x + y)) = M((x_n - x) + (y_n - y))$$

$$\leq M(x_n - x) + M(y_n - y)$$

Since $M(x_n - x) \to 0$ and $M(y_n - y) \to 0$

Then $M((x_n + y_n) - (x + y)) \to 0$ as $n \to \infty$

Then $x_n + y_n \to x + y$.

2. Let $x_n \to x$

$$M(cx_n - cx) = M(c(x_n - x)) = M(x_n - x)$$

Since $M(x_n - x) \to 0$ as $\to \infty$ , then $M(cx_n - cx) \to 0$ as $n \to \infty$

Then $cx_n \to cx$ .

Journal of Iraqi Al-Khwarizmi Society (JIKhS) Volume:3 Issue:Special 2019 pages: 1-7

**Theorem (2.14) :**

Let $X, Y$ be a modular spaces and $T: X \to Y$ is linear operator .Then $T$ is compact linear operator if and only if it maps every bounded sequence $\{x_n\}$ in $X$ onto a sequence $\{T(x_n)\}$ in $Y$ which has a convergent subsequence.

**Proof :**

If $T$ is compact linear operator and $\{x_n\}$ is bounded ,then the closure of $\{T(x_n)\}$ in $Y$ is compact and from definition (2.6) shows that $\{T(x_n)\}$ contains a convergent subsequence.

Conversely ,suppose that every bounded sequence $\{x_n\}$ contains a subsequence

$\{x_{n_k}\}$ such that $\{T(x_{n_k})\}$ converges in $Y$.Considerevery bounded subset $A \subset X$,and let $\{y_n\}$ be any sequence in $T(A)$ .Then $y_n = T(x_n)$ for some $x_n \in A$, and $\{x_n\}$ is bounded

since $A$ is bounded . By a ssumption , $\{T(x_n)\}$ contains a convergent subsequence.

Hence $\overline{T(A)}$ by definition (2.12) because $\{y_n\}$ in $T(A)$ was arbitrary. By definition , this shows that $T$ is compact linear operator.

**Theorem (2.15):**

Let $X$ and $Y$ be a modular spaces and $T_g : X \to Y$ is compact linear operator where $g = 1,2$ .Then $T_1 + T_2$ is compact linear operator and also $cT_g$ is compact linear operator, where $c$ any scalar $c \in F - \{0\}$, ( $F$ is field and $g = 1,2$ ).

**Proof**

Let $\{x_n\}$ bounded sequence in modular space $X$

Since $T_g : X \to Y$ is compact linear operator where $g = 1,2$

Then from theorem (2.14) we have $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that

$\{T_1(x_{n_k})\}$ and $\{T_2(x_{n_k})\}$ are converges in $Y$

then from theorem (2.13) we have $\{T_1(x_{n_k}) + T_2(x_{n_k})\}$ is converges in $Y \Longrightarrow \{(T_1 + T_2)(x_{n_k})\}$ is converges in $Y$

Therefore from theorem (2.14 ) we have $T_1 + T_2$ is compact linear operator .

Also since $\{T_g(x_{n_k})\}$ is converges in $Y$ where $g = 1,2$.

Then by theorem (2.13) $\{cT_g(x_{n_k})\}$ is converges in $Y$ where where $c$ any scalar

$c \in F - \{0\}$, ( $F$ is field)

Then from theorem (2.14) we have $cT_g$ is compact linear operator .

**Theorem (2.16) :( Riesz Lemma )**

Let $C$ be a closed proper subspace of modular space and Let $\lambda$ a real numbers such that $0 < \lambda < 1$ then there exists a vector $x_\lambda \in X$ such that $M(x_\lambda) > 0$ and $M(x - x_\lambda) \geq \lambda$ for all $x \in C$ .

**Proof:**

Since $C$ be a closed proper subspace of $X \Longrightarrow C \neq X$

There exist $x_0 \in X$ such that $x_0 \notin C$

Let $d = \inf\{M(x - x_0) : x \in C\}$

Since $x_0 \notin M \Longrightarrow d > 0$ , since $0 < \lambda < 1 \Longrightarrow \frac{d}{\lambda} > d$

By the definition of infimum, there exist $x_1 \in C$ such that $d < M(x - x_1) \leq \frac{d}{\lambda}$

Let $x_\lambda = K(x_0 - x_1)\, where\, K = M(x_0 - x_1)^{-1} > 0$

Then $M(x_\lambda) = M\big(K(x_0 - x_1)\big) = M(x_0 - x_1) > 0$

Let $x \in C \Longrightarrow k^{-1}x + x_1 \in C$

$M(x - x_\lambda) = M\big(x - k(x_0 - x_1)\big)$

$= kM(k^{-1}x + x_1) - x_0) \geq kd$

We have $\frac{1}{k} \leq \frac{d}{\lambda}$ so $kd \geq \lambda$

Hence $M(x - x_\lambda) \geq \lambda$

**Theorem (2.17) :**

Let $X$ be a modular space and assume that $A = \{x : M(x) = 1\}$ is compact then $X$ is finite dimensional .

**Proof:**

Suppose that $X$ is not finite dimensional

Choose $x_1 \in A$ and let $M_1$ be the subspace spanned by $x_1$

Then $M_1$ is proper subspace of $X$

Since $M_1$ is finite dimensional $\Longrightarrow M_1$ is complete $\Longrightarrow M_1$ is closed

By Riesz Lemma then there exist $x_2 \in A$ such that $M(x_2 - x_1) \geq \frac{1}{2}$

Let $M_2$ be the closed proper subspace of $X$ generated by $\{x_1, x_2\}$

Then $\exists x_3 \in A$ such that $\mu(x_3 - x) \geq \frac{1}{2}$

It follows that neither the sequence $\{x_n\}$ nor its any subsequence converges , this is contradiction

Then $X$ is finite dimensional.

**Lemma (2.18) :**

Let $T: X \to X$ be a compact linear operator and $S: X \to X$ be a bounded linear operator on a modular spaces $X$ .Then $ST$ are compact linear operator.

**Proof:**

To prove that $ST$ is compact linear operator

Let $(x_n)$ be any bounded sequence in $X$

Then $(Tx_n)$ has convergent subsequence $(Tx_{nk})$ by theorem ( 2.14 ) and $(STx_{nk})$ converges by theorem (2.15 )Then $ST$ is compact (by theorem( 2.15)) .

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