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Type of Family Sets With Some of the their Properties

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Abstract. In this work, we introduced new concepts in rings and fields such as (α- ring is α-σ-ring and β-σ-ring is β-ring) and studied the properties of each of them and the relationships between them, in addition to that, the relationships between these concepts and the previous concepts, and we studied a previous one by developing a new definition λ – system to get some results that deal with these definitions used [see 7].

Keyword: rings , fields ,α-σ-ring , α-ring , β-σ-ring , β-ring, λ –System.

1. Introduction

The concept of rings and fields was introduced by Robert[9] as a family of subsets of a given set, for the purpose of studying and developing the measure as a set function. As we note in Hallos [2] for sound algebraic reasons for using the terms "lattice" and " ring ".

For certain categories of setes - reasons that are more convincing than the similarities made Hausdorff use "ring" and "field". The notion of ring was studied by Jan Derezinski [4] and Paul [6], The concept of σ–field was studied by Dietmar[1] and Robret[9]. The concept of α–field, α– σ–field, β–field ,β– σ–field was studied by Ibrahim and Hassan[3]. They provided basic properties, descriptions, and examples of these concepts.

2. Rings and Fields

In this section we will introduce some types of rings and fields as a family of subsets of a given set and present some of the main properties and results related to rings and fields with illustrative examples.

Definition 2. 1 [5]

A nonempty family $\mathcal{F}$ of subsets of a set Ω is called:

1. A semiring on Ω if :

   $a.$ If A,B $\in \mathcal{F}$, then A ∩ B $\in \mathcal{F}$.

   $b.$ If A , B $\in \mathcal{F}$, then $A/B = \bigcup_{i=1}^{n} A_i$, where A_i disjoint sets in $\mathcal{F}$.

2. A semifield on Ω if :

   $a.$ If A , B $\in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

   $b.$ If A $\in \mathcal{F}$, then $A^c = \bigcup_{i=1}^{n} A_i$, where A_i are disjoint sets in $\mathcal{F}$.

3. A ring on Ω if :

   $a.$ If $A, B \in \mathcal{F}$, then $A/B \in \mathcal{F}$.

   $b.$ If $A, B \in \mathcal{F}$, then $A \, U \, B \in \mathcal{F}$

4. A field on Ω if ,

   $a.$ If $A \in \mathcal{F}$, Then $A^c \in \mathcal{F}$.

   $b.$ If $A_1 , A_1 , \dots A_n \in \mathcal{F}$, Then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$.

5- A σ-ring on Ω if :

a. If A,B ∈ $\mathcal{F}$, then A/B ∈ $\mathcal{F}$.

b. If A_n ∈ $\mathcal{F}$ n=1,2,… , then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

6. A σ- field on Ω if :

a. If $A \in \mathcal{F}$, Then $A^c \in \mathcal{F}$.

b. If $A_1, A_2, \ldots \in \mathcal{F}$, Then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Example 2.2

1. Let $\Omega = \{a, b, c\}$, $\mathcal{F}_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \Omega\}$ and $\mathcal{F}_2 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are semiring and semifields on Ω but that $\mathcal{F}_1 \cap \mathcal{F}_2$ fails to be a semiring and semifields on Ω.
2. The family $\mathcal{F}$ of all bounded subsets of ℝ is a ring on ℝ.
3. The family $\mathcal{F}$ of all subsets of a set Ω is a field on Ω .
4. The family $\mathcal{F}$ of all countable subsets of a set Ω is a σ-ring on a set Ω .
5. The family $\mathcal{F}$ of all countable subsets of a set Ω and their complements is a σ- field on Ω .

Theorem 2. 3[5]

1. Every semifield is a semiring, but the converse is not true.
2. Every field is ring but the converse is not correct.
3. Every ring is also a semiring, but the converse is not correct.
4. Every field is also semifield , but the convers not correct.
5. Every σ-ring is also ring but the convers not true.
6. Every σ-field is a field but the converse is not correct.
7. Every σ-field is σ-ring but the converse is not correct .

Example 2. 4

1. Let $\mathcal{F}$the family of all bounded subsets of ℝ is a ring on ℝ . But not field because ℝ∉ Ω , ℝ not bounded set.
2. The set $\mathcal{F}$ of all open intervals (a, b) ,closed of intervals$[a,b]$ and half open intervals $[a, b), (a, b]$, including the empty interval $(a.a) = \emptyset$ and the single-element set $[a, a] = \{a\}$, is a semiring on $\mathcal{F}$.But not a ring for it is not closed under the operation of difference.
3. The family $\mathcal{F} = \{(a, b] : -\infty < a \leq b < \infty\}$ is a semifield but not field on Ω
4. Let Ω=ℝ and $\mathcal{F}$ consist of all finite disjoint union or right-semi closed intervals. then $\mathcal{F}$ is a ring. but $\mathcal{F}$ is not a σ-ring. Because if we take A_n= (0 , 1- (1/n)] , n= 1,2,… then A_n ∈ F but $\bigcup_{n=1}^{\infty} A_n$ = (0 ,1) ∉ $\mathcal{F}$
5. Let Ω = ℝ , and take $\mathcal{F}$ to be the family of all finite disjoint union of intervals of the form $(a, b] = \{x \in \mathbb{R}: a \leq x \leq b\}$ .By convention we also count (a, ∞) as right semi closed . $\mathcal{F}$ is an a field but not σ-field on Ω . Because $A_n = (0,1 - \frac{1}{n}] \in \mathcal{F}$ for all $n = 1,2, \ldots$, but $\bigcup_{n=1}^{\infty} A_n = (0,1) \notin \mathcal{F}$
6. Let Ω ={1,2,3} and $\mathcal{F}$ ={ ø ,{2},{3},{2,3}, Ω } then it is clear that $\mathcal{F}$ is σ-ring but not

σ- field . Because {2} ∈ $\mathcal{F}$, but $\{2\}^c$ = { 1, 3 } ∉ $\mathcal{F}$

Definition 2. 5

Let $\mathcal{F}$ be a non-empty collection of subsets of a set Ω. Then $\mathcal{F}$ is called

1. A α-ring if the following conditions are satisfied :

$a.\ \emptyset \in \mathcal{F}$.

$b.$ If $A_1, A_1, \dots A_n \in \mathcal{F}$ then $\bigcup_{i=1}^{n} An \in \mathcal{F}$.

2. A α-field if the following conditions are satisfied:[3]

$a.\ \Omega \in \mathcal{F}$.

$b.$ If $A_1, A_1, \dots A_n \in \mathcal{F}$ then $\bigcup_{i=1}^{n} An \in \mathcal{F}$.

Example 2.6

1. Let $\Omega = \{a, b, c, d\}$ and $\mathcal{F}$ ={{b,c},{b,d},{b,c,d},$\emptyset$ }, then $\mathcal{F}$ is a α-ring of a set Ω.

2. Let $\Omega = \{a, b, c\}$ and $\mathcal{F} = \{\{a\}, \{b\}, \{a, b\}, \Omega\}$, then $\mathcal{F}$ is a α-field of a set Ω.

Remark 2.7

A α-field $\not\Leftrightarrow$ a α-ring , because Ω ∈ α-field ,bot not necessary Ω ∉ α-ring and ∅ ∈ α-ring , bot not necessary ∅ ∉ α-field.

Theorem 2.8

1. Every ring is α-ring but the converse is not correct.

2. Every σ-ring is α-ring, but the converse is not correct.

3. Every field is α-field but the converse is not correct.[3]

4. Every σ-field is α-field, but the converse is not true.[3]

Proof :

1. Let $\mathcal{F}$ be a ring of a set Ω. We have $\emptyset \in \mathcal{F}$.

Let $A_1, A_2, \dots A_n \in \mathcal{F}$, since $\mathcal{F}$ is ring , we have $\bigcup_{i=1}^{n} An \in \mathcal{F}$, hence $\mathcal{F}$ is a α-ring of a set Ω.

2. Let $\mathcal{F}$ is σ-ring on set Ω , Let $\emptyset \in \mathcal{F}$, if A,B $\in \mathcal{F}$ then A/B $\in \mathcal{F}$

Let $A_1, A_2, \dots \in \mathcal{F}$ consider $A_k = \emptyset$ for al $k > n$, since $\mathcal{F}$ *is* σ-ring,Then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, but $A_k = \emptyset$ for all $k > n$, then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{k=1}^{n} A_k$, for all $k > n$. Hence $\bigcup_{k=1}^{n} A_k \in \mathcal{F}$, therefore $\mathcal{F}$ is α-ring of a set Ω.

Conversely, the above theorem is not true. The next example explain that.

Example 2.9

In example (2.6) part (1) indicate that Ω is α-ring but not a ring and not σ-ring , because $\{b, c\}, \{b, d\} \in \mathcal{F}$, but $\{b, c\} / \{b, d\} = \{c\} \notin \mathcal{F}$.

Definition 2.10

Let $\mathcal{F}$ be a non-empty collection of subsets of a set Ω. Then $\mathcal{F}$ is called

1. A α-σ-ring if the following conditions are satisfied :

$a.\ \emptyset \in \mathcal{F}$.

$b.$ If $A_1, A_2, \dots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

2. A α-σ-field if the following conditions are satisfied : [3]

$a.\ \emptyset, \Omega \in \mathcal{F}$.

$b.$ If $A_1, A_2, \dots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Example 2.11

1. Let $\Omega = \{a, b, c, d\}$ and $\mathcal{F} = \{\{a, c\}, \{a, d\}, \{a, c, d\}, \emptyset\}$ then $\mathcal{F}$ is α-σ-ring of a set Ω.

2. Let $\Omega = \{a, b, c, d\}$ and $\mathcal{F} = \{\Omega, \{a, c\}, \{a, d\}, \{a, c, d\}, \emptyset\}$ then $\mathcal{F}$is α-σ-field of a set Ω.

Theorem 2.12

1. Every σ-ring is α-σ-ring , but the converse is not correct.

2. Every σ-field is α-σ-field , but the converse is not correct.[3]

3. Every α-σ-ring is α-ring, but the converse is not correct.

4. Every α-σ field is α-field, but the converse is not correct.[3]

Proof :

1. Let $\mathcal{F}$ is σ-ring on Ω ,then by definition σ-ring, we have $\emptyset\in \mathcal{F}$. Let $A_1 , A_2 , \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, hence $\mathcal{F}$ is α-σ-ring of a set Ω.

3. Let $\mathcal{F}$ is α-σ-ring on Ω ,then by definition α-σ-ring , we have $\emptyset \in \mathcal{F}$.

Let $A_1 , A_2 , \ldots A_n \in \mathcal{F}$ and consider $A_i = \emptyset$ for all $i > n$, since $\mathcal{F}$ is α-σ-ring ,then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, but $A_i = \emptyset$ for all $i > n$, then $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{n} A_i$, for all $i > n$.

Hence $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$, therefore $\mathcal{F}$ is α-ring of a set Ω.

Conversely, the above theorem is not true. The next example explain that.

Example 2. 13

1. Let $\Omega = \{ a, b, c, d \}$ and $F = \{ \{a, c\}, \{a, d\} , \{a, c, d\} , \emptyset \}$ then $\mathcal{F}$ is a α-σ-ring but not σ – ring of a set Ω . Because $\{a, c\} / \{a, d\} = \{c\} \notin \mathcal{F}$

2. In example (2.6) show that. $\mathcal{F}$ is a α-ring but not α-σ-ring . Because $\Omega \notin \mathcal{F}$.

Proposition 2. 14

Let $\{\mathcal{F}_i , i\in\wedge\}$ be a family of α-σ-ring of a set Ω . then $\bigcap_{i\in\wedge} \mathcal{F}_i$ is α-σ-ring of a set Ω.

Proof :

Since $\mathcal{F}_i , i\in\wedge$ is α-σ-ring of a set Ω . Then $\emptyset\in \mathcal{F}_i , i\in\wedge$, hence $\emptyset \bigcap_{i\in\wedge} \mathcal{F}_i$.

Let $A_1 , A_2 , \ldots \in\bigcap_{i\in\wedge} \mathcal{F}_i$, then $A_1 , A_2 ,\ldots \in \mathcal{F}_i , i\in\wedge$.

Since $\mathcal{F}_{i\,i}$ ∀i∈∧ is α – σ- ring, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}_i , i\in\wedge$.

Hence $\bigcup_{k=1}^{\infty} A_k \in \bigcap_{i\in\wedge} \mathcal{F}_i$, therefore $\bigcap_{i\in\wedge} \mathcal{F}_i$ is α-σ-ring of a set Ω.

The next example indicate that the union of two α-σ-ring of a set Ωis not necessarily a α-σ-ring of a set Ω .

Example 2. 15

Let Ω= $\{ a, b, c, d \}$ and $\mathcal{F}_1 = \{ \{a \}, \{a, d\}, \emptyset \}$and $\mathcal{F}_2 = \{ \{ c\}, \{ c, d\} , \emptyset \}$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are α-σ-ring of a set Ω , but

$\mathcal{F}_1 \cup \mathcal{F}_2 = \{ c\}, \{ c, d\} , \{a \}, \{a, d\}, \emptyset \}$ is not α-σ-ring of a set Ω, since $\{a\}, \{c\} \in \mathcal{F}_1 \cup \mathcal{F}_2$, but $\{a \} \cup\{c\} = \{a, c \} \notin \mathcal{F}_1 \cup \mathcal{F}_2$.

Proposition 2. 16 [3]

Let $\{\mathcal{F}_i , i\in\wedge\}$ be a family of α-σ- field of a set Ω .then $\bigcap_{i\in\wedge} \mathcal{F}_i$ is α-σ- field of a set Ω.

And indicate that the union of two α-σ- field of a set Ωis not necessarily a α-σ- field of a set Ω.

Remark 2.17

A α-ring definition equivalence β-field if the following conditions are satisfied : [3]

a. $\emptyset \in \mathcal{F}$.

b. If $A_1 , A_2 , \ldots A_n \in \mathcal{F}$ then $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$.

Example 2. 18

1. Let $\Omega = \{ a, b, c \}$ and $\mathcal{F} = \{ \emptyset, \{a\}, \{a, c\}, \{c\} \}$ then $\mathcal{F}$ is β-ring of a set Ω

2. Let $\Omega = \{ a, b, c \}$ and $\mathcal{F} = \{ \emptyset, \{a\}, \{a, c\} \}$ then $\mathcal{F}$ is β-field of a set Ω.

Theorem 2.19

1. Every ring is β-ring, but the converse is not correct
2. Every σ-ring is β-ring, but the converse is not true.
3. Every field is β-ring, but the converse is not true
4. Every field is β-field , but the converse is not correct.[3]
5. Every σ- field is β-field , but the converse is not correct.[3]
6. Every ring is β-field, but the converse is not correct.[3]

Proof :

1. Let $\mathcal{F}$ is ring on Ω ,then by definition ring, we have $\emptyset \in \mathcal{F}$. Let $A_1, A_2, \ldots A_n \in \mathcal{F}$, Then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$, to prove $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$

$\bigcap_{i=1}^{n} A_i = A / \bigcup_{i=1}^{n}(A/A_i)$ and $\bigcup_{i=1}^{n}(A/A_i) \in \mathcal{F}$, therefore $A / \bigcup_{i=1}^{n}(A/A_i) \in \mathcal{F}$.

Now , $A / \bigcup_{i=1}^{n}(A/A_i) = A / \bigcup_{i=1}^{n}(A \cap A_i^c) = A \cap [\bigcup_{i=1}^{n}(A \cap A_i^c)]^c$

$= A \cap [\bigcap_{i=1}^{n}(A^c \cup A_i)] = A \cap [A^c \cup (\bigcap_{i=1}^{n} A_i)]$

$= (A \cap A^c) \cup (A \cap (\bigcap_{i=1}^{n} A_i)) = \bigcap_{i=1}^{n} A_i$

Implies that $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$. Therefore $\mathcal{F}$ is a β-ring.

2. Let $\mathcal{F}$ be a field of a set Ω . Then by definition of $\mathcal{F}$, we get $\emptyset \in \mathcal{F}$,

let $A_1, A_2, \ldots A_n \in \mathcal{F}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$, from Demorgan laws we have

$\bigcap_{i=1}^{n} A_i = (\bigcup_{i=1}^{n} A_i^c)^c$, but $A_1, A_2, \ldots A_n \in \mathcal{F}$.

Since $\mathcal{F}$ is closed under complementation, then $A_1^c, A_2^c, \ldots A_n^c \in \mathcal{F}$ and $\bigcup_{i=1}^{n} A_i^c \in \mathcal{F}$, hence $(\bigcup_{i=1}^{n} A_i^c)^c \in \mathcal{F}$. Therefore $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$, then is β-ring of a set Ω.

Conversely, the above theorem is not true. The next example explain that.

Example 2.20

1. Let $\Omega = \{ a, b, c \}$ and $\mathcal{F} = \{ \emptyset, \{c\}, \{a, c\}, \{b, c\} \}$ then $\mathcal{F}$ is β-ring of a set Ω but not ring of a set Ω and not σ-ring , Because $\{a, c\} \setminus \{b, c\} = \{a\} \notin \mathcal{F}$

2. Let $\Omega = \{ a, b, c \}$ and $\mathcal{F} = \{ \emptyset, \{c\}, \{a, c\}, \{b, c\} \}$ then $\mathcal{F}$ is β-ring of a set Ω but not field of a set Ω. Because$\{c\} \in \mathcal{F}, \{c\}^c = \{a, b\} \notin \mathcal{F}$.

Definition 2.21

Let $\mathcal{F}$ be a nonempty collection of subsets of a set Ω. Then $\mathcal{F}$ is called

1. A β-σ-ring if the following conditions are satisfied :

   $a.$ $\emptyset \in \mathcal{F}$.

   $b.$ $A_1, A_2, \ldots A_n \in \mathcal{F}$ then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

2. A β-σ-field if the following conditions are satisfied : [3]

   $a.$ $\Omega, \emptyset \in \mathcal{F}$.

   $b.$ $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Example 2.22

1. Let $\Omega = \{ a, b, c, d \}$ and $\mathcal{F} = \{ \{c\}, \{b, c\}, \{a, c\}, \emptyset \}$ then $\mathcal{F}$ is β-σ-ring of a set Ω.

2. Let $\Omega = \{ a, b, c, d \}$ and $\mathcal{F} = \{ \Omega, \{a\}, \{a, c\}, \{c\}, \emptyset \}$ then F is β- σ-field of a set Ω.

Theorem 2.23

1. Every σ-field is β-σ-ring, but the converse is not correct.

2. Every σ- ring is β-σ-ring, but the converse is not correct.

3. Every β-σ-ring is β- ring , but the converse is not correct.

4. Every σ- ring contain ø is β- σ-field, but the converse is not true.[3]

5. Every σ-field is β-σ-field, but the converse is not true.[3]

6. Every β-σ-field is β-field , but the converse is not true.[3]

Proof :

1. Let $\mathcal{F}$ is σ-field on Ω ,then by definition σ-field , we have $\emptyset \in \mathcal{F}$.

Let $A_1, A_1, \ldots \in \mathcal{F}$, Then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, to prove $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$, from demorgan laws we have $\bigcap_{i=1}^{\infty} A_i = (\bigcup_{i=1}^{\infty} A_i^c)^c$, but $A_1, A_1, \ldots A_n \in \mathcal{F}$. Since $\mathcal{F}$ is closed under complementation , then $A_1^c, A_2^c, \ldots \in \mathcal{F}$, and $\bigcup_{i=1}^{\infty} A_i^c \in \mathcal{F}$, hence $(\bigcup_{i=1}^{\infty} A_i^c)^c \in \mathcal{F}$. Therefore $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$, then $\mathcal{F}$ is β-σ-ring of a set Ω.

2. Let $\mathcal{F}$ is σ-ring on Ω ,then by definition σ-ring , we have $\emptyset \in \mathcal{F}$.

Let $A_1, A_1, \ldots A_n \in \mathcal{F}$, Then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, to prove $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$,

$\bigcap_{i=1}^{\infty} A_i = A / \bigcup_{i=1}^{\infty}(A/A_i)$ and $\bigcup_{i=1}^{\infty}(A/A_i) \in \mathcal{F}$, therefore $A / \bigcup_{i=1}^{\infty}(A/A_i) \in \mathcal{F}$.

Now , $A / \bigcup_{i=1}^{\infty}(A/A_i) = A / \bigcup_{i=1}^{\infty}(A \cap A_i^c)) = A \cap [\bigcup_{i=1}^{\infty}(A \cap A_i^c)]^c$

$$= A \cap [\bigcap_{i=1}^{\infty}(A^c \cup A_i)] = A \cap [A^c \cup (\bigcap_{i=1}^{\infty} A_i)]$$

$$= (A \cap A^c) \cup (A \cap (\bigcap_{i=1}^{\infty} A_i)) = \bigcap_{i=1}^{\infty} A_i$$

Implies that $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$. Therefore $\mathcal{F}$ is a β-σ-ring.

3. Let $\mathcal{F}$ be a β-σ-ring of a set Ω. Then $\emptyset \in \mathcal{F}$. Let $A_1, A_2, \ldots A_n \in \mathcal{F}$,take $A_{n+1}, \ldots = \Omega$

then$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{n} A_i$. Since $A_i \in \mathcal{F}$ for all $i = 1, 2, \ldots$ and $\mathcal{F}$ be a β-σ-ring, then

$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$, hence$\bigcap_{i=1}^{n} A_i \in \mathcal{F}$, therefor $\mathcal{F}$ is β-ring of a set Ω.

Conversely, the above theorem is not true. The next example explain that.

Example 2.24

1. Let $\Omega = \{ a, b, c, d \}$ and $\mathcal{F} = \{ \{c\}, \{b, c\}, \{a, c\}, \emptyset \}$ then $\mathcal{F}$ is β-σ-ring of a set Ω, but not σ-field contain $\emptyset$. Because $\{a, c\} \in \mathcal{F}$, $\{a, c\}^c = \{b, d\} \notin \mathcal{F}$.

2. Let $\Omega = \{ a, b, c, d\}$ and $\mathcal{F} = \{ \{c\}, \{b, c\}, \{a, c\}, \emptyset \}$ then $\mathcal{F}$ is β-σ-ring of a set Ω, but not σ-ring. Because $\{a, c\} / \{b, c\} = \{a\} \notin \mathcal{F}$.

3. Let $\Omega = \{ a, b, c \}$ and $\mathcal{F} = \{ \emptyset, \{a\}, \{a, c\}, \{c\} \}$ then $\mathcal{F}$ is β-ring of a set Ω but not

β-σ-ring, Because $\emptyset \notin \mathcal{F}$.

Theorem 2.25

Let $\{\mathcal{F}_i, \forall i \in \wedge\}$ be a family of β-σ-ring of a set Ω . then $\bigcap_{i \in \wedge} \mathcal{F}_i$ is β-σ-ring of a set Ω.

Proof :

Since $\mathcal{F}_i, \forall i \in \wedge$ is β-σ-ring of a set Ω . Then $\emptyset \in \mathcal{F}_i, \forall i \in \wedge$, hence$\emptyset \in \bigcap_{i \in \wedge} \mathcal{F}_i$.

Let $A_1, A_1, \ldots A_n \in \bigcap_{i \in \wedge} \mathcal{F}_i$, then $A_1, A_1, \ldots A_n \in \mathcal{F}_i, \forall i \in \wedge$.

Since $\mathcal{F}_i, \forall i \in \wedge$ is β-σ-ring, then $\bigcap_{k=1}^{\infty} A_k \in \mathcal{F}_i, \forall i \in \wedge$. Hence $\bigcap_{k=1}^{\infty} A_k \in \bigcap_{i \in \wedge} \mathcal{F}_i$, therefore $\bigcap_{i \in \wedge} \mathcal{F}_i$ is β-σ-ring of a set Ω.

Definition 2.26

Let $\mathcal{F}$ be a non-empty set . A non-empty collection $\mathcal{F} \subseteq$P($\mathcal{F}$) is said to be

1. A δ-ring of a set Ω. the following conditions are satisfied :

 a. $\Omega \in \mathcal{F}$.

 b. If $A, B \in \mathcal{F}$ $A \subset B$ then $B|A \in \mathcal{F}$.

 c. If $A_1, A_2, \ldots \in \mathcal{F}$ and $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

2. δ-Field of a set Ω. the following conditions are satisfied : [3]

 a. $\emptyset \in \mathcal{F}$

 b. If a nonempty set in $\mathcal{F}$ and $A \subset B \subseteq \Omega$, then $B \in \mathcal{F}$

 c. If $A_1, A_2, \ldots \in \mathcal{F}$ and $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

Remark 2.27 [3]

For any δ-Field of a set Ω , the following hold :

1. $\Omega \in \mathcal{F}$

2. If $A_1, A_2, \ldots A_n \in \mathcal{F}$ then $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$

3. If A_λ for some λ , then $\bigcup_{\lambda \in \wedge} A_\lambda$.

Example 2.28

1. Let $\Omega = \{a, b, c\}$ and $\mathcal{F} = \{\emptyset, \{a, b\}, \{b, c\}, \Omega\}$ .Then $\mathcal{F}$ is a δ-ring of set Ω.

2. Let $\Omega = \{a, b, c, d\}$ and $\mathcal{F} = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \Omega\}$ .Then $\mathcal{F}$ is a δ-field of set Ω.

Theorem 2.29

Every δ-field is δ-ring , but the converse is not true.

Proof:

1. Let $\emptyset \in \mathcal{F}$, since $\emptyset \subseteq \Omega$, then $\Omega \in \mathcal{F}$.

2. Let A a nonempty set on $\mathcal{F}$, A⊆ Ω then $\Omega \in \mathcal{F}$ such that Ω /A $\in \mathcal{F}$.

3. Let If $A_1, A_2, \ldots \in \mathcal{F}$ and , Since F is δ-field , then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Hence $\mathcal{F}$ is δ-ring.

Conversely, the above theorem is not true. The next example explain that.

Example 2.30

Let Ω ={ 1,2,3,4 } and $\mathcal{F}$ ={ {1,2}, {1,2,3} , {1,2,4}, Ω } . then $\mathcal{F}$ is δ-ring of a set Ω but not δ-field, Because $\emptyset \notin \mathcal{F}$. The following diagrams illustrate the relationships between the concepts used in this thesis.

Remark 2.31

Directly from the definition δ-ring we can prove each of the following:

1. Every δ-ring is α-ring, but the converse is not correct.

2. Every δ-ring is α-σ-ring, but the converse is not correct.

3. Every δ-ring is β- ring , but the converse is not correct.

4. Every δ-ring is β- σ -ring, but the converse is not true.

3. π- system and λ – system

This section, we found some new definitions such as π- system and λ – system to obtain some results deals with these definitions .We give basic properties and examples of these concepts.

Definition 3.1[7]

"A family $\mathcal{F}$ of subsets of a set Ω is named :

1. π- system on a set Ω, if A,B∈ $\mathcal{F}$, then A∩ B∈ $\mathcal{F}$.
2. λ - system (or Dynkin system) on a set Ω , if
   $a.$ $\Omega \in \mathcal{F}$
   $b.$ If A,B∈ $\mathcal{F}$ and A ⊆ B then B| A∈ $\mathcal{F}$
   $c.$ If { A_n } is an increasing sequence of sets in Ω , Then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

A λ - measurable Space is a pair $(\Omega, \mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a λ -system on Ω.

Proposition 3.2[7]

Let $(\Omega, \mathcal{F})$ be a λ-measurable space . Then $A^c \in \mathcal{F},$ if A ∈ $\mathcal{F}$. Hence ∅∈$\mathcal{F}$.

Remark 3.3[7]

If $\mathcal{F}$ is λ – system on Ω , it is clear to show that:

1. If $A_1, A_2, \ldots A_n \in \mathcal{F}$ then$\bigcup_{n=1}^{n} A_n \in \mathcal{F}$ *and* $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$
2. If $A_n \in \mathcal{F}$, $n = 1,2, \ldots$ then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Example 3.4

Every semifield is a π- system.

Example 3.5

Every σ-field $\mathcal{F}$ is a λ-system . But a field need not be a λ - system.

Theorem 3.6 [8]

"A family $\mathcal{F}$ of subsets of a set Ω is λ-system on a set Ω , iff

1. $\Omega \in \mathcal{F}$.
2. If A ∈ $\mathcal{F}$ then $A^c \in \mathcal{F}$.
3. If { A_n } is a sequence of disjoint sets in $\mathcal{F}$, then Then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$."

Theorem 3.7 [8]

"A family $\mathcal{F}$ of subsets of a set Ω is σ-field iff it is both a π- system and a λ-system on Ω."

Theorem 3.8 [8]

"Let $\mathcal{F}$ be a λ-system on a set Ω . Fix D∈$\mathcal{F}$, then $\mathcal{G}$={A∈$\mathcal{F}$: A⋂D ∈$\mathcal{F}$ }is λ-system."

Theorem 3.9 [5]

1. If $\{\mathcal{F}_a\}_{a\in\wedge}$ is a family of π-system, then $\mathcal{F} = \bigcap_{a\in\wedge} \mathcal{F}_a$ is also π- system.
2. If $\{\mathcal{F}_a\}_{a\in\wedge}$ is a family of λ-system, then $\mathcal{F} = \bigcap_{a\in\wedge} \mathcal{F}_a$ is also λ-system.
3. If $\{\mathcal{F}_a\}_{a\in\wedge}$ is a family of σ-field, then $\mathcal{F} = \bigcap_{a\in\wedge} \mathcal{F}_a$ is also σ– field.

Definition 3.10 [10]

"Let $\mathcal{G}$ be a family of subsets of a set Ω .

1.The smallest π- system including $\mathcal{G}$ named the π- system generated by $\mathcal{G}$ and it is mean by π ($\mathcal{G}$).
2.The smallest λ-system including $\mathcal{G}$ named the λ-system generated by $\mathcal{G}$ and it is mean by λ ($\mathcal{G}$).
3.The smallest σ-field including $\mathcal{G}$ named the σ-field generated by $\mathcal{G}$ and it is mean by σ ($\mathcal{G}$)".

Example 3.11

A field $\mathcal{F}$ need not be a λ-system.

Theorem 3.12

Let $\mathcal{F}$ be a λ-system on a set Ω . Fix $D \in \mathcal{F}$, then $\mathcal{F} = \{A \in \mathcal{F} : A \cap D \in \mathcal{F}\}$is λ-system.

Theorem 3.13
Let $\mathcal{F}$ is π-system then λ($\mathcal{F}$) = σ($\mathcal{F}$)
Proof :
Let $\mathcal{F}$ be a π-system on Ω , then is σ-field on Ω , $\mathcal{F}$ ⊂ σ($\mathcal{F}$) .
σ($\mathcal{F}$) is λ-system on Ω , we have λ($\mathcal{F}$) ⊂ σ($\mathcal{F}$)…(1)
Now, to show that σ($\mathcal{F}$) ⊂ λ($\mathcal{F}$) , when $\mathcal{F}$ is π-system .
Step 1 : $\mathcal{F}$ ⊂ λ($\mathcal{F}$)
σ($\mathcal{F}$) ⊂ σ(λ($\mathcal{F}$))…(2)
Step 2 : We need to show that , σ(λ($\mathcal{F}$)) = λ($\mathcal{F}$) , i.e. we must show that λ($\mathcal{F}$) is σ-field on Ω .
Let $\mathcal{F}$ be family of subsets of Ω , then $\mathcal{F}$ is σ-field iff $\mathcal{F}$ is field
$A_n \in \mathcal{F}$, $A_n \subset A_{n+1}$ $\forall n$, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, thus it is ciosed under intersection .
Let $\lambda_1(\mathcal{F})$ = { A : A ∈ λ($\mathcal{F}$) , A⋂B ∈ λ($\mathcal{F}$) ∀ B ∈ $\mathcal{F}$}
$\lambda_1(\mathcal{F})$ is a λ-system and $\mathcal{F}$ is π-system , $\mathcal{F} \subset \lambda_1(\mathcal{F})$ → λ($\mathcal{F}$) ⊂ $\lambda_1(\mathcal{F})$
But $\lambda_1(\mathcal{F})$ ⊂ λ($\mathcal{F}$) → $\lambda_1(\mathcal{F})$ = λ($\mathcal{F}$)
Let $\lambda_2(\mathcal{F})$ = { A : A ∈ λ($\mathcal{F}$) , A⋂B ∈ λ($\mathcal{F}$) ∀ B ∈ λ($\mathcal{F}$) }
$\lambda_2(\mathcal{F})$ is a λ-system and $\mathcal{F} \subset \lambda_2(\mathcal{F})$⊂ λ($\mathcal{F}$) , $\lambda_2(\mathcal{F})$= λ($\mathcal{F}$) ,
i.e. λ($\mathcal{F}$) it is ciosed under intersection , then λ($\mathcal{F}$) $\mathcal{F}$ is σ-field
σ(λ($\mathcal{F}$)) = λ($\mathcal{F}$) …from (2) σ($\mathcal{F}$) ⊂ σ(λ($\mathcal{F}$))
i.e. σ($\mathcal{F}$) ⊂ λ($\mathcal{F}$) ….(3) ⟹Thus λ($\mathcal{F}$) = σ($\mathcal{F}$).

Theorem 3. 14
Every ring is π-system but the converse is not true.

Example 3.15
Let Ω ={1,2,3} and $\mathcal{F}$ ={∅, {2}, Ω} Thus $\mathcal{F}$ is π-system but not ring .
Because Ω /{2}= { 1,3 } ∉ $\mathcal{F}$.

Theorem 3.16[7]
Let Ω_1 and Ω_2 be ≠ 0 sets, and $f: \Omega_1 \to \Omega_2$ is any function .
1.If $\mathcal{F}$ is a λ-system on Ω_1, then $\mathcal{H}$={ $A \subseteq \Omega_2$:$f^{-1}(A) \in \mathcal{F}$} is a λ-system on Ω_2.
2.If $\mathcal{G}$ is a a λ-system on Ω_2, then $f^{-1}(\mathcal{G})$ ={ $f^{-1}(A)$: $A \in \mathcal{G}$} is a λ-system on Ω_2.

Theorem 3.17[7]
If $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary family of λ-system on a set Ω where Λ ≠ ∅ , then $\mathcal{F} = \bigcap_{\lambda\in\Lambda} \mathcal{F}_\lambda$ is a λ-system on Ω.

Definition 3.18[7]
Let $\mathcal{G}$ be a family of sets of a set Ω . The smallest λ-system Containing $\mathcal{G}$ called the λ-system generated by $\mathcal{G}$ and it is Denoted by λ($\mathcal{G}$).

Theorem 3.19[7]
Let Ω_1 and Ω_2 be nonempty sets, and $f: \Omega_1 \to \Omega_2$ is any function .If $\mathcal{G}$ is a family of set in Ω_2. Then $\lambda(f^{-1}(\mathcal{G})) = f^{-1}(\lambda(\mathcal{G}))$,Where $f^{-1}(\mathcal{G})$ ={ $f^{-1}(A)$: $A \in \mathcal{G}$}.

Definition 3.20[7]
Let $\mathcal{G}$ be a family of subsets of a set Ω, and let $A \subset \Omega$. The restriction (or trace) of $\mathcal{G}$ on A is the collection of all sets of the form $A \cap B$, were $B \in \mathcal{G}$, and it is denoted by $\mathcal{G}_A$(or $A \cap \mathcal{G}$)

$$\mathcal{G}_A = A \cap \mathcal{G} = \{ A \cap B : B \in \mathcal{G}\}$$

$\mathcal{G}_A$ is family of subsets of A. The fuzzy λ-system$\sigma(\mathcal{G}_A)$ generated by $\mathcal{G}_A$. Some time denoted by $\lambda_a(A \cap \mathcal{G})$, i.e. $\lambda(\mathcal{G}_A) = \lambda_a(A \cap \mathcal{G})$.

Theorem 3.21[7]
Let $\mathcal{G}$ be a family of subsets of a set Ω, and let $A \subset \Omega$.
Then $A \cap \lambda(\mathcal{G}) = \lambda(A \cap \mathcal{G})$.

References

[1] Dietmar, A. S. 2016. Measure and Integration. https://people.math.ethz.ch/~salamon/PREPRINTS/measure.pdf
[2] Hallos, P.R. ,"Measure Theory", New York, N.Y.: D. Van Nostrand Co. 1950.
[3] Ibrahim , S.A. and Hassan H. E. , " On α–field and β–field", IOP Conf. Series: Journal of Physics: Conf. Series 1294 (2019) 032015.
[4] Jan Derezinski. 2017. Measure Theory. www.fuw.edu.pl/~derezins/mat-m.pdf
[5] Noori, F. A.: Measure Theory and Applications. Deposit number in the National Library and Archives in Baghdad (1780). First edition, Iraq (2018) .
[6] Paul, R. H. 1974. Measure Theory. Springer – Verlag New York.
[7] Rasha A. H. and Noori F. A. , " λ - Borel and λ - Baire Credibility on Normal Topological Spaces" , College of Science, University of Al-Qadisiyah, Iraq.
[8] Rasha A.H. and Noori F.A.: Credibility Space and Fuzzy Variables on λ –System , Journal of Physics: Conference Series(Research under publication) 2021.
[9] Robert, B. A., " Real Analysis and Probability ", (1972) Academic Press, INC, University of Illinois .
[10] Sharma J. N., Vishnu K., "Topology", Krishna Prakashan Media P Ltd, 2003.