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Solution of Systems of Nonlinear Weakly-Singular Volterra Integral Equations Via Backward Differentiation Techniques Numerically

Kameran Ali Ameen[1] and Borhan F. Jumaa[2] and Abbas H. Taqi[3]

[1, 2]Computer Science Department, College of Computer Science and Information Technology, University of Kirkuk, Kirkuk, Iraq.

3Department of Mathematics / College of Science / University of Kirkuk

[1]kameran.ameen@gmail.com

[2]borhan_nissan@yahoo.com

[3]dr.altaqi@yahoo.com

Abstract: This paper uses Backward Differentiation Formulas of step (1, 2, 3, 4, 5 and 6) to evaluate a system of three and two second kind of nonlinear Weakly-Singular Volterra integral equations numerically. The system is solved by using MATLAB 2014b software. Finally, a number of examples are proposed to demonstrate the accuracy and effectiveness of this formula.

Keywords: Nonlinear integral equation, Weakly-singular Volterra, Backward differentiation formulas.

1 Introduction:

The following equation formulates the general nonlinear weakly-singular Volterra integral equations of the second kind:

$$\varphi(x) = g(x) + \int_0^x \frac{\beta}{[f(x)-f(t)]^\alpha} G\big(\varphi(t)\big)dt \quad , \; 0 < \alpha < 1 \, , \; x \in [0,T] \qquad \ldots(1)$$

where β is a constant, and $G(\varphi(t))$ is a nonlinear function of $\varphi(t)$. In the literature, one can find these equations in several mathematical chemistry and physics, such as heat conduction, stereology, radiation of heat from a semi-infinite solid and crystal growth. It is also assumed at the function $g(x)$ is a given real valued function [1, 2].

The books edited by Wazwaz [3] and Linz [4] contain some different methods to solve the system of nonlinear weakly-Singular Volterra integral equations analytically. The author of [5] used the Trapezoidal Predictor-Corrector Method to solve the system of two nonlinear Volterra Integral Equations of the Second Kind. Furthermore, the authors in [6] proposed a numerical solution of System of two nonlinear Volterra integral equations. Moreover, Borhan and Abbas in [7] used non-Polynomial spline method to solve systems of two nonlinear Volterra integral equations. Finally, the authors in [8] used predictor-corrector methods as a numerical solution of these systems.

Accordingly, this work studies the system of two and three nonlinear weakly-singular Volterra integral equations of the second kind. The following form formulates the unknown functions that appear inside and outside the integral sign [8].

$$\varphi_i(x) = g_i(x) + \sum_{j=1}^{m} \int_0^x k_{ij}(x,t) G_{ij}(\varphi_j(t))dt \qquad i = 1,2,\ldots,n \qquad \ldots(2)$$

where $\varphi_i(x)$ are unknown functions, and $G_{ij}(\varphi_j(t))$ is a nonlinear function of $\varphi_j(t)$. The functions $g_i(x), i = 1, 2, 3$ and kernels $k_{ij}(x,t), 1 \le i, j \le 3$ are given real-valued functions on subsets of R^3 and R^1, respectively. The kernels k_{ij} comprises of singular kernels where the generalized form is as follows:

$$k_{ij} = \frac{1}{[f(x) - f(t)]^{\alpha_{ij}}} \quad , \qquad 1 \le i, \qquad j \le 3$$

2 Backward Differentiation Formulas [2]:

Let $Q_q(t)$ denote the polynomial of degree $\le q$ that interpolate Y(t) at the points $t_{n+1}, t_n, \dots, t_{n-q+1}$ for some $q \ge 1$,

$$Q_q(t) = \sum_{j=-1}^{q-1} Y(t_{n-j}) l_{j,n}(t), \qquad \dots(3)$$

where $\{l_{j,n}(t): j = -1, \dots, q-1\}$ are the Lagrange interpolation basis functions for the nodes $t_{n+1}, t_n, \dots, t_{n-q+1}$. Use

$$Q'_q(t_{n+1}) = Y'(t_{n+1}) = f(t_{n+1}, Y(t_{n+1})) \qquad \dots(4)$$

Combining (4) with (3) and solving for $Y(t_{n+1})$, we obtain

$$Y(t_{n+1}) = \sum_{j=0}^{q-1} \alpha_j Y(t_{n-j}) + h\beta g(t_{n+1}, Y(t_{n+1}))$$

The p-step Backward Differentiation Formulas is given by

$$y_{n+1} = \sum_{j=0}^{q-1} \alpha_j y_{n-j} + h\beta g(t_{n+1}, y_{n+1}) \qquad \dots(5)$$

The truncation error for (5) can be obtained from the error formulas for numerical differentiation

$$T_n(Y) = -\frac{\beta}{q+1} h^{q+1} Y^{(q+1)}(\rho_n) \text{ for some } t_{n-q+1} \le \rho_n \le t_{n+1} \qquad \dots(6)$$

3 Numerical Methods

For simplicity, an interval [a,b] is taken to develop the numerical method for approximation solution of a system of the type (2). Accordingly, a grid of $N+1$ equally spaced points $x_i = a + ih, i = 0,1, \dots, N$ is defined, where $h = \frac{b-a}{N+1}$. Now, the following represents the p-step Backward formula for each i^{th} segment.

$$y_{n+1} = \sum_{j=0}^{q-1} \alpha_j y_{n-j} + h\beta g(t_{n+1}, y_{n+1}) \qquad \dots(7)$$

To solve the equation (2) we combine (7) with (2), then we obtain:

$$\varphi_i(x_{n+1}) = g_i(x_{n+1}) + \sum_{j=0}^{q-1} \alpha_j \varphi_{n-j} + h\beta \sum_{j=1}^{m} k_{ij}(x_{n+1}, t_{n+1}) G_{ij}(\varphi_j(t_{n+1})) \qquad \dots(8)$$

Accordingly, the system of two and three nonlinear Weakly-Singular Volterra integral equations of the second kind can be approximated via (8). The approximation is defined as NLWSVIEBD algorithm, which is stated as follows:

Algorithm (NLWSVIEBD):

Step 1: Set $h = \frac{b-a}{N}$; $x_i = x_0 + ih$, i = 0,1,2, ... N, x_i = a, x_N = b.

Step 2: For $q = 1$ to 6 perform the next steps.

Step 3: For $i = 1\ to\ N + 1$ perform the next steps.

Step 4: Set $\varphi_i(x_0) = g_i(x_0)$.

Step 5: Evaluate $\varphi_i(x_{n+1})$ using equation (8).

4 Numerical Example:

In this section, we suggest a number of examples to illustrate the methods of Section 3.

4.1 Example (1)

System of two nonlinear Weakly-Singular Volterra integral equations of the second kind is defined as follows:

$$\varphi(x) = x^2 - \frac{2}{5}x^{\frac{5}{2}} - \frac{3}{14}x^{\frac{14}{3}} + \int_0^x \left(\frac{1}{(x^5 - t^5)^{\frac{1}{2}}}\varphi^2(t) + \frac{1}{(x^7 - t^7)^{\frac{1}{3}}}\emptyset^2(t) \right) dt$$

$$\emptyset(x) = x^3 - \frac{4}{15}x^{\frac{15}{4}} - \frac{5}{28}x^{\frac{28}{5}} + \int_0^x \left(\frac{1}{(x^5 - t^5)^{\frac{1}{4}}}\varphi^2(t) + \frac{1}{(x^7 - t^7)^{\frac{1}{5}}}\emptyset^2(t) \right) dt$$

which has the exact solution: $\left(\varphi(x), \emptyset(x)\right) = (x^2, x^3)$

Table (1): The truncation error of the numerical solution via (**NLWSVIEBD**) algorithm.

q	Error in φ	Error in $\emptyset$
1	0.192913E-02	0.313335E-02
2	0.479822E-03	0.653059E-03
3	0.119803E-03	0.142067E-03
4	0.299413e-04	0.634997E-04
5	0.748472E-05	0.976963E-05
6	0.187687E-05	0.778953E-05

4.2 Example (2):

System of two nonlinear Weakly-Singular Volterra integral equations of the second kind is defined as follows:

$$\varphi(x) = sin^{\frac{1}{2}}x + \frac{3}{2}(cosx - 1)^{\frac{2}{3}} - \frac{3}{2}sin^{\frac{2}{3}}x + \int_0^x \left(\frac{1}{(cosx - cost)^{\frac{1}{3}}}\varphi^2(t) + \frac{1}{(sinx - sint)^{\frac{1}{3}}}\emptyset^2(t) \right) dt$$

$$\emptyset(x) = cos^{\frac{1}{2}}x + 3(cosx - 1)^{\frac{1}{3}} + 3sin^{\frac{1}{3}}x + \int_0^x \left(\frac{1}{(cosx - cost)^{\frac{2}{3}}} \varphi^2(t) - \frac{1}{(sinx - sint)^{\frac{2}{3}}} \emptyset^2(t) \right) dt$$

which has the exact solution: $\big(\varphi(x), \emptyset(x)\big) = (sin^{\frac{1}{2}}x \ , \ \ cos^{\frac{1}{2}}x)$

Table (2): The truncation error of the numerical solution via (**NLWSVIEBD**) algorithm.

Q	Error in φ	Error in $\emptyset$
1	0.768544E-01	0.617599E-01
2	0.185549E-01	0.156383E-01
3	0.435448E-02	0.598975E-02
4	0.110156E-02	0.778439E-03
5	0.274531E-03	0.237855E-03
6	0.688776E-04	0.553324E-04

4.3 Example (3):

System of three nonlinear Weakly-Singular Volterra integral equations of the second kind is defined as follows:

$$\varphi(x) = e^x - \frac{1}{2}(e^{4x} - 1)^{\frac{1}{2}} - \frac{1}{3}(e^{6x} - 1)^{\frac{1}{2}} + \int_0^x \left(\frac{1}{(e^{4x} - e^{4t})^{\frac{1}{2}}} \emptyset^2(t) + \frac{1}{(e^{6x} - e^{6t})^{\frac{1}{2}}} \omega^2(t) \right) dt$$

$$\emptyset(x) = e^{2x} - \frac{1}{3}(e^{6x} - 1)^{\frac{1}{2}} - (e^{2x} - 1)^{\frac{1}{2}} + \int_0^x \left(\frac{1}{(e^{6x} - e^{6t})^{\frac{1}{2}}} \omega^2(t) + \frac{1}{(e^{2x} - e^{2t})^{\frac{1}{2}}} \varphi^2(t) \right) dt$$

$$\varphi(x) = e^{3x} - (e^{2x} - 1)^{\frac{1}{2}} - \frac{1}{2}(e^{4x} - 1)^{\frac{1}{2}} + \int_0^x \left(\frac{1}{(e^{2x} - e^{2t})^{\frac{1}{2}}} \varphi^2(t) + \frac{1}{(e^{4x} - e^{4t})^{\frac{1}{2}}} \emptyset^2(t) \right) dt$$

which has the exact solution: $(\varphi(x), v(x), w(x)) = (e^x, e^{2x}, e^{3x})$

Table (3): The truncation error of the numerical solution via (**NLWSVIEBD**) algorithm.

q	Error in φ	Error in $\emptyset$	Error in ω
1	0.582415E-02	0.412917E-01	0.015870E-00
2	0.182492E-02	0.209369E-01	0.828769E-01
3	0.719683E-03	0.529830E-02	0.232478E-01

4	0.357568E-03	0.914723E-03	0.651729E-02
5	0.117323E-03	0.467798E-03	0.885640E-03
6	0.443456E-04	0.010317E-03	0.336518E-03

5 Conclusion:

In this paper, we proposed using of Backward Differentiation Formulas method to solve the system of two and three nonlinear Weakly-Singular Volterra integral equations of the second kind numerically. The findings demonstrated the superiority of our proposed methods achieved more better accuracy in solving the systems. Furthermore, q is the key to getting good approximation results, where the increasing of q leads to an increase in the number of points, and the error approaches zero.

الملخص

في هذا البحث, استخدمنا صيغ التمايز الى الوراء من الخطوات (1,2,3,4,5,6) لحل منظومة مكونة من معادلتين وثلاث معادلات فولتيرا المفرد ضعيف التكاملية غير الخطية من النوع الثاني وقد استخدمنا برنامج matlab14 لحل النظام. وأخيرا، قدمنا العديد من الأمثلة التوضيحية لإظهار فعالية ودقة هذه الصيغ.

References

[1] J.R.Rise, Numerical Method Software and Analysis, macgra Hill, 1985.

[2] K.Atkinson,W. Han,D.Stewart , Numerical Solution of Ordinary Differential Equations, United States of America, 2009.

[3] A.M.Wazwaz, Linear and Nonlinear Integral Equation, Methods and Applications, Springer, 2011.

[4] P.Linz, Analytical and numerical methods for Volterra Equations, studies in Applied Mathematics, SIAM, Philadelphia, 1985.

[5] F.J.Borhan, Solving the System of Two Nonlinear Volterra Integral Equations of the Second Kind Using the Trapezoidal Predictor-Corrector Method, British Journal of Mathematics & Computer Science , vol.20, No.3, P.P. 1-10, 2017.

[6] A. M.Dalal, Numerical Solution of System of Two Nonlinear Volterra Integral Equations, International Journal of Computers and Technology, Vol.12, No.10, P.3967-3975, 2014.

[7] F.J.Borhan, and H.T.Abbas, Non-Polynomial Spline Method for the Solution of the System of two Nonlinear Volterra Integral Equations, Kirkuk University Journal/Scientific Studies(KUJSS) , Vol.11, No.3, P.P.(15-25), 2016.

[8] F.J.Borhan, and H.T.Abbas, Using Predictor-Corrector Methods for Numerical Solution of System of Non-linear Volterra Integral Equations of Second Kind, AL-Rafiden Journal of Computer Sciences and Mathematics, Vol.6, No.3, P.P.(93-109), 2009.