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Semi-approximately 2-Absorbing Sub-module and Semi-approximately 2-Absorbing Module

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ABSTRACT

In this paper we define and study new concept , denoted by semi-approximately 2-absorbing submodule of M if whenever $a \in R$, $m \in M$ and $a^3m \in N$ implies that either $a^2m \in N$ or $a^3 \in (N :_R M)$ And M is called Semi-approximately 2-absorbing Module if zero sub-module is Semi-approximately 2-absorbing sub-module. As generalization to semi 2-absorbing submodule. Many properties and examples are introducing of this concept.

1. Introduction

Throughout this paper R commutative ring with identity and M unitary R-module. It is well known a proper sub-module N of M is called prime sub-module $rx \in N, r \in R, x \in M$ implies that $x \in N$ or $r \in (N: M)$[1]. Where $N: M)=\{r\in R: rM \leq N\}$. As a generalization of prime sub-module semi-prime sub-module N is called semi-prime sub-module if whenever $, r \in R, , x \in M$ with $r^2x \in N$ implies that $, rx \in N$[2]. N is called 2-absorbing sub-module if whenever $a, b \in R, m \in M$ and $abm \in N$, then either $am \in N$ or $ab \in (N: M)$[3]. As a generalization of 2-absorbing submodule in [4] N is called semi-2-absorbing submodule if whenever $a \in R, m \in M$ and $a^2m \in N$ implies that either $am \in N$ or $a^2 \in (N: M)$. This led us introduce the concept semi- approximately 2-absorbing submodule of M if whenever $a \in R$, $m \in M$ and $a^3m \in N$ implies that either $a^2m \in N$ or $a^3 \in (N :_R M)$ and semi-approximately 2-absorbing module . We provide many properties, characterizations and relationship between semi-approximately 2-absorbing and other concepts.

2. Semi- approximately 2-absorbing submodule

In this section we define new concepts and study some properties and relatives with other classes of submodules

Definition 2.1: A proper submodule N of R-module M is called semi- approximately 2-absorbing submodule of M if whenever $a \in R$, $m \in M$ and $a^3m \in N$ implies that either $a^2m \in N$ or $a^3 \in (N :_R M)$.

A proper ideal I of a ring R is called semi- approximately 2-absorbing ideal if whenever $a , b \in R$ and $a^3b \in I$ implies that either $a^2b \in I$ or $a^3 \in I$

Remarks and Examples 2.2:

(1) It is clear that every semi 2-absorbing submodule is a semi approximately -2-absorbing .

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Proof: Let $a^3m \in N$ so $a^2(am) \in N$,put am= m' get am= $m' \in N$, $a^2(m') \in N$ we get $a(m') \in N$ or $a^2 M \subseteq N$ so $a(am) \in N$ or $a^3\ M \subseteq N$ then $a^2m \in N$ or $a^3\ M \subseteq N$. But the converse is not true, for example:

-Consider in the Z-module Z_8 Let N=(0) and $2^3.1 = 0 \in N$, $2^2.1 \notin N$,but 2^3Z_8 =(0) $\subseteq$ N so N is a semi- approximately 2-absorbing but

$2^2.2 = 0 \in N$ and 2.2=4$\notin$ N and 2^2Z_8 =4Z_8 $\nsubseteq$ N.

-Consider in the Z-module 36Z is not semi 2-absorbing sub module since: $3^2.4 = 12 \in N$,but 3.4 = 12 $\notin$36Z and $3^2 \notin (36Z : Z) = 36Z$.

(2) Every semi-prime submodule is a semi approximately 2-absorbing submodule. But the converse is not true, for example:

(0) in the Z-module Z_4 is a semi- approximately 2-absorbing submodule of Z_4, but (0) not semi-prime since 2.2.1 = 0 but 2.1 ≠ 7

(3) It is clear that every approximately 2-absorbing submodule is a semi approximately -2-absorbing

submodule. However the converse is not true in general as we shown in

the following example: Consider Z⊕ Z as Z-module and N=6Z ⊕ (0) a submodule of Z⊕Z but N is not approximately 2-absorbing submodule by Examples (1.2.1)part (2). But N is a semi-approximately 2-absorbing submodule, since if $a^3(m_1,0) \in 6Z \oplus (0)$, then, $a^3m_1 \in 6Z$ but it is clear that 6Z is a semi-prime. So $a^2m_1 \in 6Z$ and; that is

$a^2(m_1\ ,0)$)∈ 6Z ⊕ (0) = N. Thus N is a semi approximately -2-absorbing.

(4) Every quasi-prime submodule is a semi approximately -2-absorbing submodule. But the converse is not true in general for example :

(0) in the Z-module Z_4 is a semi-2-absorbing submodule of Z. but (0)

not quasi-prime since 2.2.1 =0 but 2.1≠ 0

(5) Let N , K be a submodules of R-module M and N ⊆K. If N is a semi approximately -2 absorbing of M then N is a semi approximately -2-absorbing.

Proof : Let a ∈R, m∈K such that $a^3m \in N$. Since K<M then m ∈ M as N is a semi approximately -2-absorbing submodule of M and $a^3m \in N$ then

either $a^2m \in N$ or $a^3 \in (N:_R\ M)$.

If $a^3 \in (N :_R\ M)$. then $a^3\ M \subseteq N$, and since K⊆ Mimplies, $a^3K \subseteq a^3M$

hence $a^3K \subseteq N$, therefore $a^3 \in (N:_R\ K)$.

Thus N is a semi approximately -2-absorbing submodule of K.

proposition 2.3: Let Mbe an R-module and N submodule of M, K⊆M .Then N is a semi approximately if and only if $a^3K \subseteq N$ implies $a^2K \subseteq N$ or $a^3 \in (N: M)$.

Proof: (⇐) It is clear

(⇒) Let $a^3K \subseteq N$ Suppose there exists $x \in K$ such that $a^2x \notin N$

Since $a^3K \subseteq N$, so $a^3K \in N$ for each $k \in K$, but N is a semi approximately -2-absorbing and $a^2x \notin N$. Hence $a^3 \in (N: M)$.

Proposition 2.4: Let N be a proper submodule of an R-module M. if N is a semi approximately -2-absorbing submodule of Mthen (N: M) is a semi approximately -2- absorbing ideal.

Proof : Let $a,b \in R$ such that $a^3 b \in (N: M)$. then $a^3bM \subseteq N$ So $a^3bm \in N$ for each $m \in M$ and assume that $a^3 \in (N: M)$.

Since N is a semi approximately -2-absorbing submodule then $a^2bm \in N$ for each $m \in M$ So. $a^2b \in (N: M)$. Thus (N: M) is a semi approximately -2-absorbing ideal.

The converse of Proposition 2.4 hold under the class of multiplication modules.

Proposition 2.5: Let N be a submodule of a multiplication R-module M such that

(N:R M)is a semi approximately 2-absorbing ideal of R. Then N is semi approximately 2-absorbing submodule of M .

Proof: Let a, b $\in$ R, $m \in M$, and $a^3m \in N$ then $a^3(m) \subseteq N$. Since M is a

multiplication R-module, there exists an ideal I of R such that (m) = IM . Thus $a^3IM \subseteq N$. Hence, $a^3I \subseteq (N:RM)$. Now by assumption, (a)2I $\in$ (N :R M) or $a^3 \in (N :R M)$ Therefore $a^2I M \subseteq N$ or $a^3 \in (N :RM)$. Thus $a^2(m) \subseteq N$ or $a^3 \in (N:R M)$.thus N is semi approximately 2-absorbing submodule of M.

Corollary 2.6: Let N a submodule of cyclic R-module M. Then N is a semi approximately -2-absorbing submodule if and only if (N: M) is a semi approximately -2-absorbing ideal.

Proof : Since every cyclic module over commutative ring is a multiplication module, hence by Proposition 2.6 the result is obtained.

Proposition 2.7: Let M be a faithful finitely generated multiplication R-module, N a proper submodule of M .Then the following statement are equivalent:

(1) N is a semi approximately -2- absorbing submodule of M.

(2) (N: M) is a semi approximately -2-absorbing ideal.

(3) N =IM for some semi approximately -2-absorbing ideal I of R.

Proof: (1)⟺(2) By Proposition (2.5) and (2.6)

(2)⟹(3) It is clear

(3)⟹(1) Let $a^3m \in N$, hence $a^3(m) \subseteq N$ Since M is multiplication(m)=JM for some ideal J of R. Hence $a^3JM \subseteq IM$ as M is finitely generated faithful multiplication ,so $a^3J \subseteq I$. But I is a semi approximately -2-absorbing ideal , so either $J^2 \subseteq I$ or $a^2 \in (I{:}R)$.

by Proposition (2.4) This implies $a^2JM \subseteq IM = N$ or

$a^2 \in I = (IM{:}M) = (N{:}M)$.thus $a^2m \in N$ or $a^3 \in (N{:}RM)$.

Proposition 2.8: Let N be a proper submodule of an R-module M. The following statements are equivalent :

(1) N is semi approximately 2-absorrbing submodule of M.

(2) $(N :_M I)$ is semi approximately 2-absorbing , for each ideal I of R with $IM \not\subseteq N$

(3) $(N{:}_M (r))$ is semi approximately 2-absorbing submodule for each

$r \in R$ with $rM \not\subseteq N$

Proof: (1)⟹(2) Let I be an ideal of R with $IM \not\subseteq N$ then $(N :_M I) \neq M$

Iet a , b ∈ R , m ∈ M, then $a^3 (Im) \subseteq N$ But N is semi approximately 2-absorbing submodule of M , so by Proposition(2.1.3) , either $a^2(Im) \subseteq N$ or $a^3 \in (N : M)$. Hence either $a^2m \in (N : I)$ or $a^3 \in ((N_M{:}I){:} M)$. Thus $(N :_M I)$ is semi approximately 2-absorbing submodule .

(2)⟹(3) It is clear .

(3)⟹(1) Take r =1 then (N : (1))= N , so N is semi approximately 2-absorbing.

Proposition 2.9: Let M be an R-module , N a proper submodule of M ,if N is semi approximately 2-absorbing then $(N{:}_R \langle m\rangle)$ is semi approximately 2-absorbing ideal of R for each $m \in M - N$.

Proof: Let $a^3 b \in (N{:}_R\langle m\rangle)$ for some $m \in R$, then $(a^3b)m \in N$, but N is semi approximately 2-absorbing submodule then $a^2(bm) \in N$ or $a^3 \in (N :_R M)$,

so that $a^2 b \in (N :_R M)$ or $a^3 \in (N :_R M)$ hence $(N{:}_R (m))$ is semi approximately 2-absorbing ideal of R.

Proposition 2.10: Let N be a submodule of an R-module M. Then N is semi approximately 2-absorbing submodule of M if and only if $(N{:} a^3 m) = (N{:} a^2m)$ for each $m \in M$ or $a^3 \in (N :_R M)$.

Proof: (⟹) Suppose $a^3 \notin (N :_R M)$. . To prove $(N{:} a^3m) = (N{:} a^2m)$ It is clear that $(N{:} a^2m) \subseteq (N{:} a^3m)$.

Let $r \in (N{:} a^3m)$, then $a^3rm \in N$ and since N is semi approximately -2-absorbing and $a^3 \notin (N :_R M)$ so $a^2rm \in N$ and hence $r \in (N{:} a^2m)$. Thus $(N{:} a^3m) = (N{:} a^2m)$.

(⟸) Let a^3m N.Then $(N{:} a^3m) = R$. But $(N{:} a^3m) = (N{:} a^2m)$ or

$a^3 \in (N{:} M)$ bv hypothesis. Therefore $(N{:} a^2m) = R$ and hence $a^2 m \in N$. So either $a^2 m \in N$ or $a^3 \in (N :_R M)$. Hence N is semi- approximately 2-absorbing.

Proposition 2.11: f : M ⟶M′ be an epimorphism , such that kerf ⊆ N and N is semi approximately 2-absorbing submodule of M then f(N) is semi approximately 2-absorbing submodule of M´.

proof: Let $a^3m' \in f(N), m' \in M', a \in R$ since f is onto, m′ = f(m) for some m ∈ M then $a^3f(m) \in f(N)$ so abf (m) = f (n) , for some n ∈ N

we get $a^3m - n \in kerf \subseteq N$ implies that $a^3m \in N$ but

(N is semi approximately 2-absorbing) so either a2m ∈ N or a3 ∈ (N: M)

if a2m ∈ N Then $a^2f(m) \in f(N)$ so a2m′ ∈ f (N)

if a3 ∈ (N: M) then $a^3M \subseteq N$ and so a3f(M) ⊆ f (N)and we get

a3 ∈ (f (N): f(M)) Then f (N) is semi approximately 2-absorbing submodule of M′.

Corollary 2.12: Let N is semi approximately 2-absorbing submodule of M with K ⊆ N then $\frac{N}{K}$ is semi approximately 2-absorbing submodule of $\frac{M}{K}$.

Proof: Let $\pi: M \to \frac{M}{K}$, π is the natural epimorphism and hence ker $\pi = K \subseteq N$

then $\frac{N}{K}$ is semi approximately 2-absorbing submodule of $\frac{M}{K}$.

Proposition 2.13: Let $\varphi: M \longrightarrow M'$ be an R-epimorphism . If W is semi approximately 2-absorbing submodule of M' , then φ–1(W) is semi approximately 2-absorbing submodule of M .

proof: Let a3m ∈ φ 1(W) where a ∈ R, m∈ M,then φ (a3m) ∈ W that is a3 φ (m) ∈ W and W is semi approximately 2-absorbing then either

a2 φ (m) ∈ w or a3 ∈ (W: M') that is a2 m ∈ φ^{-1}(W) and if

$a^3 \in (W{:}M')$ *then* $a^3M' \subseteq W$ but φ (M) ⊆ M'*so* $a^3\varphi$ (M) ⊆ W that is a^3 M ⊆ φ^{-1} (W) so $a^3 \in (\varphi^{-1}(W){:}\ M)$.

then φ^{-1} (W) is semi approximately 2-absorbing submodule of M.

Corollary 2.14: Let M R-module and K ⊆ N < M if $\frac{N}{K}$ is semi approximately 2-absorbing submodule of $\frac{M}{K}$. then N is semi approximately 2-absorbing submodule of M.

Proof: Let $\pi: M \to \frac{M}{K}$, π is the canonical epimorphism, π -1($\frac{N}{K}$) is semi approximately 2-absorbing,since $\frac{N}{K}$ is semi approximately 2-absorbing submodule of $\frac{M}{K}$. but π -1($\frac{N}{K}$)= N,so N is semi approximately 2-absorbing submodule of M.

Proposition 2.15 : Let M be a module over a Principal Ideal Ring (P.I.R) R, N a proper submodule of M and I an ideal of R Then N is a semi approximately -2-absorbing submodule of M if and only if I^3m ⊆ N implies I^2m ⊆ N or I^3 ⊆ (N: M) for any ideal I of R.

Proof: (⟹) Let I be ideal of R and let m ∈M. Since R is P.I.R, then I=$< a >$ for some a∈ R. If I^3m ⊆ N then $< a >^3$m ⊆ N, therefore a^3m ∈ N which implies that a^2m ∈ N or a^3 ∈ (N: M) Thus I^2m ⊆ N or I^3 ⊆ (N: M).

(⟸)It is clear .

Proposition 2.16: Let M_1, M_2, be R-modules and M=$M_1 \oplus M_2$, and let N and W be a proper Sub Modules of $M_1$and M_2respectively. Then

1)N is semi approximately 2-absorbing in M_1 if and only if $N \oplus M_2$ is semi approximately 2-absorbing in M =$M_1 \oplus M_2$and

2)W is semi approximately 2-absorbing in M_2 if and only if $M_1 \oplus$ W is semi approximately 2-absorbing in M .

Proof: ⟹

Let a^3 $(m_1, m_2) \in N \oplus M_2$,when a∈ R and $(m_1, m_2) \in M$ then

a^3 $m_1 \in N$ and a^3 $m_2 \in M_2$ .Since N is semi approximately 2-absorbing in M_1 implies that either $a^2 m_1 \in N$ or $a^3 \in (N : M_1)$. So that

a^2 $(m_1, m_2) \in N \oplus M_2$ Or $a^3 \in (N: M_1)$ then $a^3 \in (N \oplus M_2 : M_1 \oplus M_2)$.

Hence $N \oplus M_2$ is semi approximately 2-absorbing in $M_1 \oplus M_2$.

⟸ Let a^3 $m_1 \in N$, where a∈R, $m_1 \in M_1$, then for any $m_2 \in M_2$, a^3 $(m_1, m_2) \in N \oplus M_2$.Since $N \oplus M_2$ is semi approximately 2-absorbing so either a^2 $(m_1, m_2) \in N \oplus M_2$, or $a^3 \in (N \oplus M_2 : M_1 \oplus M_2)$ =$(N: M_1)$ Then $a^2 m_1 \in N$ or $a^3 \in (N: M_1)$, N is semi approximately 2-absorbing submodule in M. The proof of (2) is similarly.

Proposition 2.17: Let N semi approximately 2-absorbing submodule of R-module

M=$M_1 \oplus M_2$ and annM_1 +annM_2=R then

(1) $N = N_1 \oplus M_2$ and N_1 is semi approximately $2-$absorbing inM_1 .

(2) $N = M_1 \oplus N_2$ and N_2 is semi approximately $2-$absorbing inM_2 .

(3) $N = N_1 \oplus N_2$ and N_1 is semi approximately $2-$absorbing in M_1 and N_2 is semi approximately $2-$absorbing inM_2.

proof: Since annM_1 +annM_2=R, then by the proof of [1,Theorem 2.4]

$N = N_1 \oplus N_2$, for some submodules N_1 of M_1 and $N_2$of M_2 .

We have

(1) $N_1 < M_1$,and $N_2 = M_2$

(2) $N_1 = M_1$, and$N_2 < M_2$,

(3) $N_1 < M_1$, and $N_2 < M_2$

Case (1) and (2): $N = N_1 \oplus M_2$ or $N = M_1 \oplus N_2$. Since N_1 and N_2 is semi approximately 2-absorbing in M_1 and M_2,so by Proposition (2.1.16),then N_1, N_2 is semi approximately $2-$ absorbing inM_1, M_2

Case (3): Let $a^3\, m_1 \in N$, where a∈R ,$m_1 \in M_1$. Then

$a^3\,(m_1, 0) \in N = N_1 \oplus N_2$. Since N is semi approximately 2-absorbing in M, then either $a^2\,(m_1,0) \in N$ or $a^3 \in (N_1 \oplus N_2 : M_1 \oplus M_2)$, so $a^2 m_1 \in N_1$

Or $a^3 \in (N_1 : M_1)$.hence $N_1$is semi approximately 2-absorbing in M_1.

Similarly we get N_2 is semi approximately 2-absorbing in M_2.

Proposition 2.18: If N_1, N_2 is semi approximately $2-$ absorbing in M_1, M_2, such that $(N_1 : M_1) = (N_2 : M_2)$.then $N = N_1 \oplus N_2$ semi approximately 2-absorbing submodule of $M = M_1 \oplus M_2$.

proof: Let $a^3\,(m_1, m_2) \in N_1 \oplus N_2$ that is $a^3 m_1 \in N_1$ and $a^3 m_2 \in N_2$.since N_1, N_2 is semi approximately $2-$ absorbing ,then $a^2 m_1 \in N_1$or

$a^3 \in (N_1 : M_1)$ and $a^2 m_2 \in N_2$ or $a^3 \in (N_2 : M_2) = (N_1 : M_1)$,so

$a^2 m_1 \in N_1$ and $a^2 m_2 \in N_2$ or $a^3 \in (N_1 : M_1)$ thus

$a^2\,(m_1, m_2) \in N_1 \oplus N_2$ or $a^3 \in (N : M)$.hence is a semi approximately -2-absorbing.

proposition 2.19: Let N is semi approximately 2-absorbing submodule of M and S multiplicative subset of R,then $S^{-1}N$ is semi approximately 2-absorbing S^{-1} R-submodule of S^{-1} M .

proof: Let $\frac{a}{s_1} \in S^{-1}$ R, $\frac{\overline{m}}{s_2} \in S^{-1}M$,then$(\frac{a}{s_1})^3 \frac{\overline{m}}{s_2} \in S^{-1}N$ then There exists t∈S such that $ta^3 m = a^3 tm \in N$ since N is semi approximately 2-absorbing so either $a^2 tm \in N$ or $a^3 \in (N: M)$ so

then $\frac{a^2 tm}{s_1 s_2 t} = \frac{a^2 m}{s_1 s_2} \in S^{-1}N$, or $\frac{a^3}{s_1} \in S^{-1}\,(N: M) \subseteq (S^{-1}N: S^{-1}M)$ then

$\frac{a^2 m}{s_1 s_2} \in S^{-1}N$ or $\frac{a^3}{s_1} \in (S^{-1}N: S^{-1}M)$.

Hence $S^{-1}N$ is semi approximately 2-absorbing.

3. Semi approximately -2-Absorbing Modules.

In this section we introduce the concept of semi approximately -2-absorbing modules. Some of properties and relationships with other classes of modules are explained.

So we give the following definition :

Definition 3.1: An R-module M is called semi approximately -2-absorbing module if (0) is a semi approximately -2- absorbing submodule of M.

Remarks and Examples 3.2:

(1) Every a semi -2-absorbing module is a semi approximately -2-absorbing module.

(2) Every semi-prime module is a semi- approximately 2-absorbing module but the converse is not true in general, for example: Z_4 as Z-module is a semi approximately -2-absorbing since (0) is a semi approximately -2-absorbing submodule of Z_4 but it is not semi-prime.

(3) Every quasi-prime module is a semi approximately -2-absorbing module. But the converse is not true in general for example: Z_4as Z-module is a semi approximately -2-absorbing module, but it is not quasi-prime since 2.2.1=0 and 2.1≠0

(4)Every submodule of semi approximately -2-absorbing module is a semi approximately -2-absorbing module.

Proposition 3.3: If M is a semi approximately -2-absorbing module, then $ann_R M$ is a semi approximately -2-absorbing ideal.

Proof: By applying Proposition (2.4) when N =(0), we get the result.

Proposition 3.4: Let M be a multiplication R-module. Then M is a semi approximately -2-absorbing module if and only if annM is a semi approximately -2-absorbing ideal.

Proof : (⇒)It follows by Proposition (3.3).

(⇐)It follows by Proposition (2.5).

Corollary 3.5: Let M be a faithful multiplication R-module. Then the following statements are equivalent:

(1) M is a semi approximately -2-absorbing module

(2) R is a semi approximately -2-absorbing ring

Proof: (1) Since M is a semi approximately -2-absorbing module, so by Proposition(3.4) annM is a semi approximately -2-absorbing ideal . But

annM =(0). Thus (0) is a semi approximately -2-absorbing ideal,that is R is a semi approximately -2-absorbing ring.

(2) R is a semi approximately -2-absorbing, so (0_R) is a semi approximately -2-absorbing, but (0_R) = $ann_R M$ since M is faithful. Thus M is a semi approximately -2-absorbing module by Proposition(3.4).

Proposition 3.6: Let M be an R-module. If M is a semi approximately -2-absorbing module, then annN is a semi approximately -2-absorbing ideal for each nonzero submodule N of M.

Proof: Let N be a nonzero submodule of M, first ann_RN ≠R because if ann_R =R, then N=(0) which is a contradiction!

Now let a^3b∈ annN for some a, b∈R. Then a^3bN =0. Since M is a semi approximately -2- absorbing module, so by Proposition (2.3) either

$a^2bN=(0)$ or $a^3 \in ((0):M)$ and hence either $a^2b \in$ annN or $a^3 \in$ annN, since annM $\subseteq$ annN.Thus annN is a semi approximately-2-absorbing ideal.

Recall that " Let R be an integral domain. A module M is called divisible if, for every $0 \neq r \in R$ then r M = M "[5]

Proposition 3.7: Over an integral domain R. Then M is a semi approximately -2-absorbing module if and only if M is a quasi-prime module.

Proof : ($\Rightarrow$)Let abm=0, where a,b$\in$R,m$\in M$

If ab=0 then a=0 or b=0, so am=0 or bm=0

If ab$\neq$0, then a $\neq$0 and b$\neq$0 since R is integral domain.

If am=0 we are done. If am $\neq$0 and a$\neq$0 and M is divisible, then

$a^2M = M$, So m=a^2m' then abm=aba^2m'= a^3bm= 0.

But (0) is semi approximately -2-absorbing implies that, either a^2b m' =0 or $a^3 \in$ann M.

If $a^3 \in$ann M then a^3M=0 but a $\neq$0 then $a^2 \neq$0.

It follows that $a^3M = M = 0$ which is a contradiction!

Therefore at $a^3 \notin$ann M. Thus abm=0, so a^2b m' =0 so bm$=0$

Thus (0) is quasi-prime.

($\Leftarrow$)It is clear.

Corollary 3.8: Let M be a nonzero divisible module over an integral domain R. The

following conditions are equivalent:

(1) M is a semi approximately -2-absorbing module.

(2) M is a quasi-prime module.

(3) M is a prime module.

Proof: (1)$\Leftrightarrow$ (2) It follows by Proposition (3.7).

(2)$\Leftrightarrow$ (3) It follows by[6,Proposition (1.5.10)]

Proposition 3.9: Let N be submodule of an R-module M. Then N

is semi approximately 2-absorbing submodule if and only if $\frac{M}{N}$

is semi approximately 2-absorbing module.

Proof: ($\Longrightarrow$)Let $a^3(x+N) = N = 0_{\frac{M}{N}}$, where a $\in$ R, x$\in$ M

Then $a^3x+N = N$, so $a^3x \in N$. Since N is semi approximately 2-absorbing,

then either $a^2x \in N$ or $a^3 \in (N:M)$,hence we get either

$a^2(x+N) = N = 0_{\frac{M}{N}}$ or $a^3 \in (0_{\frac{M}{N}} : \frac{M}{N})$ since $(N:M) = \text{ann}\,\frac{M}{N}$

Hence $\frac{M}{N}$ is semi approximately 2-absorbing module.

(⟸) let $a^3x \in N$ are a,∈R, x∈ M. Then $a^3 (x+N) = N=0_{\frac{M}{N}}$ but $0_{\frac{M}{N}}$

Is semi approximately 2-absorbing submodule,then

$a^2(x+N) = N$ or $a^3 \in \text{ann}\frac{M}{N} = (N:M)$ so that $a^2x \in N$ or $a^3 \in (N:M)$.

Hence N is semi approximately 2-absorbing submodule of M.

Proposition 3.10: An R-module M is a semi approximately -2-absorbing module if and only if either $\text{ann}a^2m=\text{ann}a^3m$ for any m∈ M such that $a^3m \neq 0$ or $a^3M=0$

Proof: (⇒)Let $r \in \text{ann}a^3m$, $a^3m \neq 0$. Then $a^3rm =0$. But M is a semi approximately -2-absorbing and $a^3 \notin \text{ann}M$, so that $a^2rm=0$; that is $r \in \text{ann}a^2m$. Thus $\text{ann}a^2m=\text{ann}a^3m$.

(⇐) It is clear.

Proposition 3.11: Let $M = M_1 \oplus M_2$ be an R-module. If M is a semi approximately -2-absorbing module, then M_1 and M_2 are semi approximately -2-absorbing module.

Proof: By Remarks and Examples 3.2 part 3 the result hold.

Theorem 3.12: Let M_1 and M_1 be prime R-modules. Then $M = M_1 \oplus M_2$ is a semi approximately -2-absorbing module.

Proof: Let $a^3(m_1, m_2)=(0,0)$ where a∈R, $(m_1, m_2) \in M$. Then $a^3m_1=0$ and

$a^3m_2=0$ that a $(a^2m_1)=0$ and $a(a^2m_2)=0$.

Since M_1 and M_1 be a prime R-module either ($a^2m_1=0$ or $a \in \text{ann}M_1$) and

($a^2m_2=0$ or $a \in \text{ann}M_2$).

(1) If $a \in \text{ann}M_1$ and $a \in \text{ann}M_2$, then $a \in annM_1 \cap annM_2 = annM$ but $a \in annM$ so $a^3 \in annM$.

(2) If $a^2m_1=0$ and $a^2m_2=0$, then $a^2(m_1, m_2)=0$.

Thus M is a semi approximately -2- absorbing module.

Note: As an application of theorem (3.12), each of the following Z-module semi-2-absorbing modules : $Z_p \oplus Z_q, Z_p \oplus Z_p, Z_p \oplus Z, Q \oplus Z, Z \oplus Z, Q \oplus Q$

where p, q are two prime numbers.

Proposition 3.13: Let M $=M_1\oplus M_2$ be an R-module such that $annM_1 = annM_2$. Then M semi approximately -2-absorbing module if and only if $M_1$and M_2 are semi approximately -2-absorbing modules.

Proof: (⇐)Let $a^3(m_1, m_2)$=(0,0), where a ∈R, $(m_1, m_2) \in M$

a^3m_1=0 and a^3m_2=0. Since M_1 and M_2 are semi approximately -2-

absorbing modules, then either (a^2m_1=0 or a^3 ∈annM_1)and

(a^2m_2=0 or a^3 ∈ann$M_2 = annM_1$).

It follows that (a^2m_1=0 and a^2m_2=0) or a^3 ∈annM_1

Hence $a^3(m_1, m_2)$= (0,0) or a^3 ∈ann$M_1 = annM_1 \cap annM_2 = annM$.

Thus (0,0) is a semi approximately -2-absorbing so M semi approximately -2-absorbing module.

(⇒) It is clear.

Proposition 3.14: For an R-module M The following assertions are equivalent:

(1) M is semi approximately 2-absorbing R-module

(2) $\mathrm{ann_M}$I is semi approximately 2-absorbing for each ideal I of R with I⊈ annM

(3) $\mathrm{ann_M}$ (r) is semi approximately 2-absorbing for each ideal r ∈ R with r ∉ annM.

proof: It follows directly by Proposition (2.8) and definition of semi approximately -2-absorbing module.

Proposition 3.15: Let M is semi approximately 2-absorbing comultiplication R-module. Then every proper submodule N of M is semi approximately 2-absorbing submodule .

proof : Let N be proper submodule of M. Let $\mathrm{ann_R N = I}$ where I is an ideal of R

So $\mathrm{ann_M ann_R}$N=N,then N= $\mathrm{ann_M}$I. since if, I ⊆ $\mathrm{ann_R}$M

So $\mathrm{ann_R}$N =$\mathrm{ann_R}$M. It follows N = $\mathrm{ann_M ann_R}$N = $\mathrm{ann_M ann_R}$M = M, then N = M which is a contradiction! Then by Proposition (3.14),

$\mathrm{ann_M}$I =N is semi approximately 2-absorbing sub module.

Proposition 3.16: Let S is multiplicative subset of R and M is an R-module,If M is a semi-approximately 2-absorbing module, then S^{-1} M is a semi approximately -2-absorbing module.

Proof: It follows by Proposition (2.19)

Lemma 2.2.17: Let M be an R-module and let $A, B < M$. Then $A = B \Leftrightarrow A_p = B_p$, for every maximal idea P of R.[21]

Corollary 3.18: Let M be a finitely generated R-module. if M_p is a semi approximately -2-absorbing R_p-module for each P maximal ideal of R, then M is a semi approximately -2-absorbing R-module.

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