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Approximate the Solution of Moisture Content Equation by a Pade Approximation

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ABSTRACT

In this paper, the solution of nonlinear three dimensional partial differential equation that's describe a mathematical formulation of soil moisture is approximated by Pade approximation to improve the convergence interval of the power series

1. Introduction

Mathematical model plays an important role in described natural and engineering problems. Due to this importance, researchers have been interesting to find efficient and accurate solutions to their problems. One of these problems is the soil moisture. In the unsaturated areas of the soil, the water occupies a part of the vacuum inside the soil, and this is called the soil moisture or the amount of water in the soil. Sometimes, soil moisture is important to estimate the irrigation requirements [1], in grain storage and food processing [2] and at tillage on soil strength in maize production [3]. Moisture content changes over time, so the amount of water flowing through the soil is erratic. So, the formation model arises as a nonlinear three- dimensional partial differential equation. The solutions of such equations are very important to many branches of engineering, such as irrigation, sanitary, civil and hydrologic.

This phenomenon have been discussed by many researchers from different perspectives, such as solving the one-dimensional flow equations in unsaturated materials by iterative methods (see [4]). Verma [5] studied one dimensional groundwater equation and used Laplace transformation method to solve this equation. However, Mehta [6] transformed the equation in such a way that the approximation method could be applied to solve it. In [7], Mehta and Patel have studied one-dimensional Groundwater equation with initial and boundary conditions with proviso spreading in Porous Media. Crank-Nicolson finite difference scheme was suggested by Borana, et al. with appropriate initial and boundary conditions to solve this equation (1). Shah and Singh [8] used q-homotopy analysis method to solve that equation.

There are many techniques and methods that have been used to solve differential equations, some of this method is used to find the exact solution for a specific problem such as separation of variables and the integral transforms (Laplace transform, Fourier transform and it.), other methods are used to find a numerical solution such as finite element method, finite difference method and it., other methods have been used to provide an analytical approximation solution for linear and nonlinear differential problems, such as the variational iteration method (VIM), homotopy perturbation method (HPM) [9, 10], combine LA- transform with decomposition method [11] and Adomian decomposition method (ADM) [12-16]. Many classes of differential equations have been solved using this decomposition methods.

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2. The Main Problem

Alaa in [17] derived a new model for the moisture content and used it to build three - dimensional groundwater flow equation as following:

$$\frac{\partial\theta}{\partial t}+\theta\frac{\partial\theta}{\partial z}=\alpha\left[\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right] \qquad (1)$$

Where θ represents moisture content in the position (x, y, z) and t is the time.

He solved (1) by an efficient method based on combines the Laplace transform and the homotopy perturbation method with appropriate initial condition:

$$f(x,y,z)=\theta(x,y,z,0)=\frac{\gamma+\beta+(\beta-\gamma)\mathrm{e}^{\mu}}{e^{\mu}+1} \qquad (2)$$

Where $\mu=\frac{\gamma}{3\,\alpha}(x+y+z)$, α, β and λ are constants. The obtained power series by this method with γ=-1 and β=1 is:

$$\begin{cases}\theta_0=\dfrac{2\,e^{\mu}}{(e^{\mu}+1)}\\ \theta_1=\dfrac{2\,e^{\mu}}{3\alpha(e^{\mu}+1)^2}\\ \theta_2=\dfrac{e^{\mu}\,(1-e^{\mu})}{9\alpha^2\,(e^{\mu}+1)^3}\\ \theta_3=\dfrac{e^{\mu}\,(1-4\,e^{\mu}+e^{2\mu})}{81\,\alpha^3\,(e^{\mu}+1)^4}\\ \theta_4=\dfrac{e^{\mu}\,(1-11e^{\mu}+11e^{2\mu}-e^{3\mu})}{972\,\alpha^4\,(e^{\mu}+1)^5}\\ \theta_5=\dfrac{e^{\mu}\,(1-26\,e^{\mu}+66\,e^{2\mu}-26\,e^{3\mu}+e^{4\mu})}{14580\,\alpha^5\,(e^{\mu}+1)^6}\\ \theta_6=\dfrac{e^{\mu}\,(1-57\,e^{\mu}+302\,e^{2\mu}-302\,e^{3\mu}+57\,e^{4\mu}-e^{5\mu})}{262440\,\alpha^6\,(e^{\mu}+1)^7}\\ \qquad\qquad \textit{And so on.}\end{cases} \qquad (3)$$

where

$$\theta(X,t)=\sum_{k=0}^{\infty}\theta_k\,t^k \qquad (4)$$

In this paper, the obtained power series is approximated by Pade approximation to improve the convergence interval of the power series.

3. Pade Approximation

Many types of problems, in various sciences as engineering and physical and it., have been successfully solved by Pade approximation [18-21]. These problems of applied sciences have been solved as an infinite power series, but it is not possible to compute the infinite series. So, the power series must be truncated at a certain term. The next step is to approximate the truncated power series by constructing a rational function matches the truncated power series as far as possible which is called Pade approximant [22]. Pade approximant approximates a truncated power series as ratio of two polynomials. The coefficients of polynomials in the numerator and denominator can be computed by matching the terms with the coefficients of truncated power series [23]. The approximation of a function using the Pade approximation is often better than the approximation by the truncated power series, because the convergence interval is greater.

Pade approximation [24-25] has been applied for rational series solutions in many areas. It is also known that Pade approximants, show superior performance over series approximations. It can be seen in many papers that Pade approximants give better numerical results than approximation by polynomial.

If the series expansion of a function $u(t)$ is as following:

$$u(t) = u_0 + u_1 t + u_2 t^2 + u_3 t^3 + u_4 t^4 + u_5 t^5 + \cdots = \sum_{k=0}^{\infty} u_k t^k \quad (5)$$

where $u_0 \neq 0$.

The symbol $[n/m]_{u(t)}$ refers to the Pade approximant for the function $u(t)$ of order $[n, m]$ which is defined by:

$$[n, m]_{u(t)} = \frac{P_n(t)}{Q_m(t)} = \frac{p_0 + p_1 t + \cdots + p_n t^n}{1 + q_1 t + \cdots + q_m t^m} \quad (6)$$

where the denominator and numerator have no common factors and $q_0 = 1$. The denominator and numerator in (6) are constructed so that $u(t)$ and $[n/m]_{u(t)}$ and their derivatives agree at $t = 0$ up to $n+m$. That is

$$u(t) - [n, m]_{u(t)} = O(t^{n+m+1}) \quad (7)$$

From (7), we have

$$u(t) \sum_{k=0}^{m} q_k t^k - \sum_{k=0}^{n} p_k t^k = O(t^{n+m+1}) \quad (8)$$

From (8), we get the following systems with $q_0 = 1$:

$$\begin{cases} u_{m+1} + u_m q_1 + \cdots + u_{m-n+1} q_n = 0 \\ u_{m+2} + u_{m+1} q_1 + \cdots + u_{m-n+2} q_n = 0 \\ \vdots \\ u_{m+n} + u_{m+n-1} q_1 + \cdots + u_m q_n = 0 \end{cases} \quad (9)$$

and

$$\begin{cases} p_0 = u_0 \\ p_1 = u_1 + u_0 q_1 \\ \vdots \\ p_n = u_n + u_{n-1} q_1 + \cdots + u_0 q_n \end{cases} \quad (10)$$

Firstly, From (9) all the coefficients q_i, $1 \leq i \leq m$ are calculated. Then, the coefficient p_i, $0 \leq i \leq n$ are determined from (10).

Note that, if the degree for numerator and denominator in (6) are equal or when the numerator has degree one higher than the denominator then the error (7) is smallest for a fixed value of $n+m+1$.

4. Approximate the Moisture Content Solution

The solution of moisture content equation (3) and (4) can be approximate by $[3,3]_{\theta(X,t)}$ as following:

$$[3,3]_{\theta(X,t)} = \frac{p_0 + p_1 t + \cdots + p_3 t^3}{1 + q_1 t + \cdots + q_3 t^3} = \theta_0 + \theta_1 t + \cdots + \theta_6\, t^6 \qquad (11)$$

Where $\theta_0, \theta_1, \ldots, \theta_6$ obtained from (3). From (3), (9), (10) and (11) we get:

$$\begin{cases} \theta_4 + \theta_3 q_1 + \theta_2 q_2 + \theta_1 q_3 = 0 \\ \theta_5 + \theta_4 q_1 + \theta_3 q_2 + \theta_2 q_3 = 0 \\ \theta_6 + \theta_5 q_1 + \theta_4 q_2 + \theta_3 q_3 = 0 \end{cases} \qquad (12)$$

and

$$\begin{cases} p_0 = \theta_0 \\ p_1 = \theta_1 + \theta_0 q_1 \\ p_2 = \theta_2 + \theta_1 q_1 + \theta_0 q_2 \\ p_3 = \theta_3 + \theta_2 q_1 + \theta_1 q_2 + \theta_0 q_3 \end{cases} \qquad (13)$$

By solving the system (12) we get:

$$\begin{cases} q_0 = 1 \\ q_1 = \dfrac{(e^{\mu} - 1)}{6\,\alpha\,(e^{\mu} + 1)} \\ q_2 = \dfrac{1}{90\,\alpha^2} \\ q_3 = \dfrac{(e^{\mu} - 1)}{3240\,\alpha^3\,(e^{\mu} + 1)} \end{cases} \qquad (14)$$

and subtitled (14) in (13) we have:

$$\begin{cases} p_0 = \dfrac{2\,e^{\mu}}{(e^{\mu} + 1)} \\ p_1 = \dfrac{e^{\mu}}{3\alpha(e^{\mu} + 1)} \\ p_2 = \dfrac{e^{\mu}}{45\alpha^2(e^{\mu} + 1)} \\ p_3 = \dfrac{e^{\mu}}{1620\alpha^3(e^{\mu} + 1)} \end{cases} \qquad (15)$$

Then subtitled (14) and (15) in (11) to obtain:

$$[3,3]_{\theta(X,t)} = \frac{\frac{2\,e^{\mu}}{(e^{\mu}+1)} + \frac{e^{\mu}}{3\alpha(e^{\mu}+1)}t + \frac{e^{\mu}}{45\alpha^2(e^{\mu}+1)}t^2 + \frac{e^{\mu}}{1620\alpha^3(e^{\mu}+1)}t^3}{1 + \frac{(e^{\mu}-1)}{6\,\alpha\,(e^{\mu}+1)}t + \frac{1}{90\,\alpha^2}t^2 + \frac{(e^{\mu}-1)}{3240\,\alpha^3\,(e^{\mu}+1)}t^3} \qquad (16)$$

By Multiplying the numerator and denominator of (16) with $(3240\alpha^3(e^{\mu} + 1))$ we have:

$$[3,3]_{\theta(X,t)} = e^{\mu}\,\frac{6480\,\alpha^3 + 1080\alpha^2 t + 72\alpha t^2 + 2t^3}{1 + 540\alpha^2(e^{\mu} - 1)t + 36\alpha(e^{\mu} + 1)t^2 + (e^{\mu} - 1)t^3} \qquad (17)$$

Where $\mu = \frac{-1}{3\,\alpha}(x + y + z)$

5. Discuss the Solution

In Figures 1, 2, 3 and 4, the solution (17), obtained by Pade approximation, of the equation (1) is illustrated numerically by taking different values of parameters and variables to compare this solution with the solution (3) that obtained by homotopy perturbation method. These cases are shown the successful use of solution approximation by Pade approximation compared to power series expansion.

5.1 Case 1: $t = 0{:}100\ , x = 1, y = 1, z = 2, \alpha = 0.2$ as in figure 1:

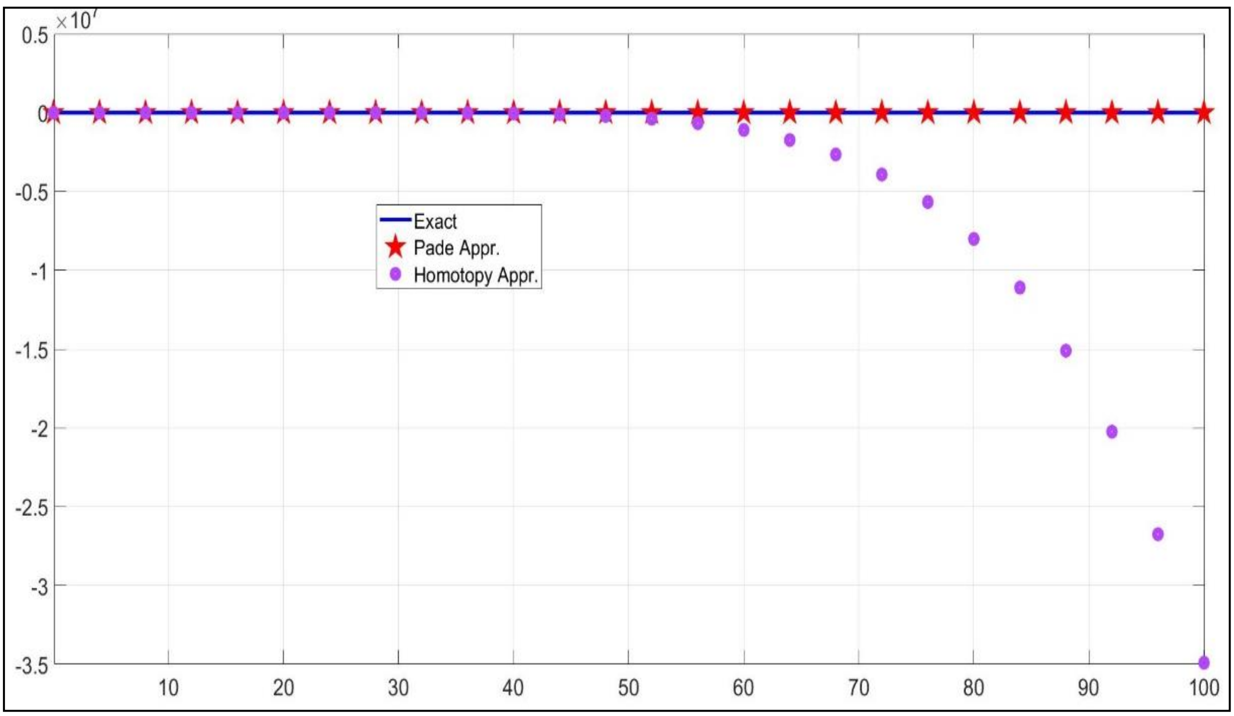

Figure 1 : Illustrate Case 1

5.2 Case 2: $t = 0{:}100\ , x = 4, y = -2, z = 2, \alpha = 0.3$ as in figure 2:

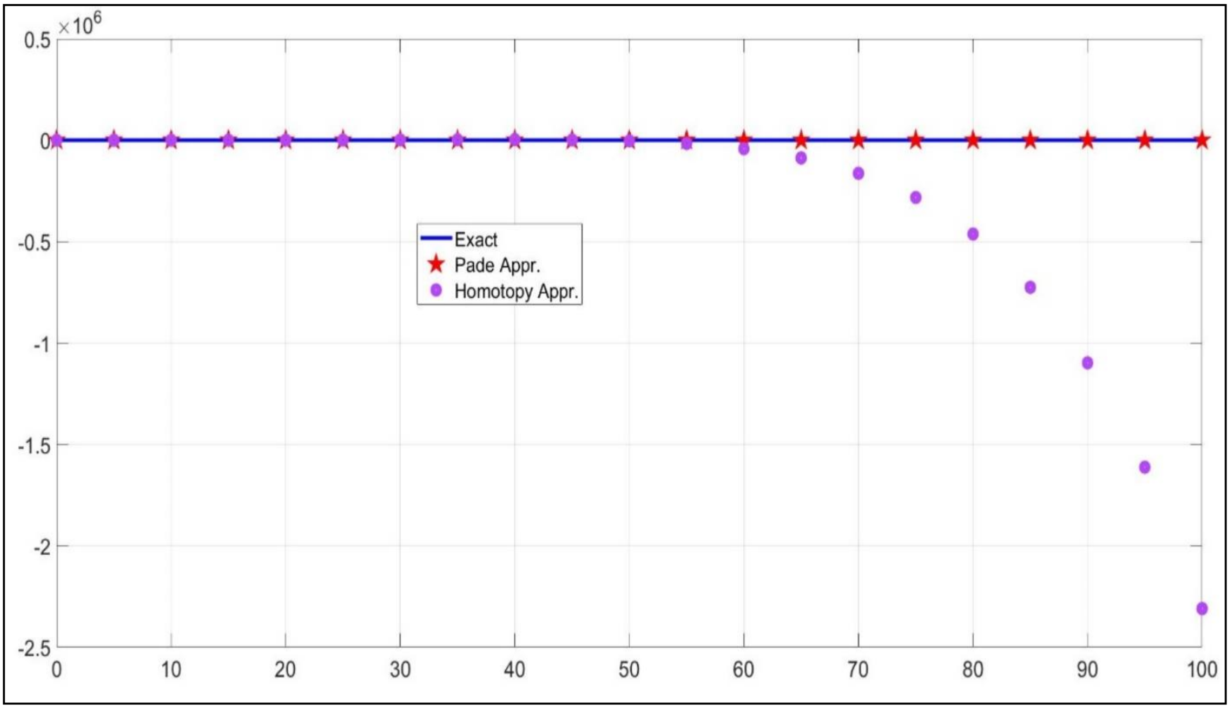

Figure 2 : Illustrate Case 2

5.3 Case 3: $t = 0{:}100$, $x = -1, y = -2, z = 5, \alpha = 0.7$ as in figure 3:

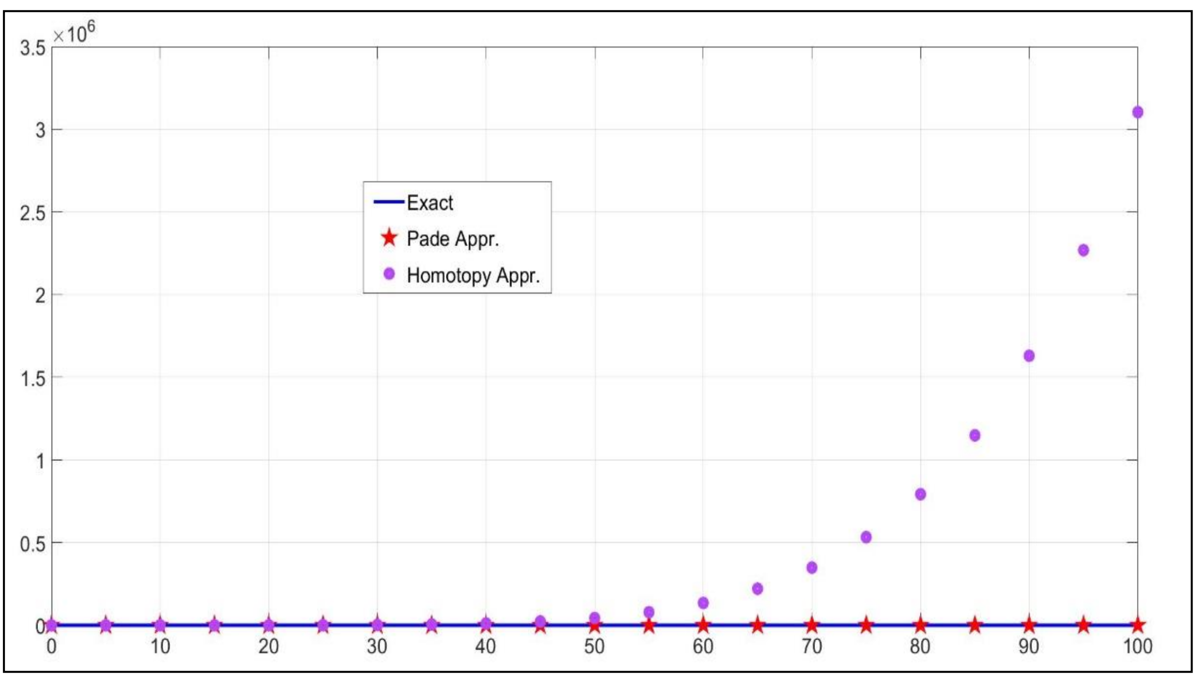

Figure 3 : Illustrate Case 3

5.4 Case 4: $t = 0{:}100$, $x = 3, y = 2, z = 5, \alpha = 0.8$ as in figure 4:

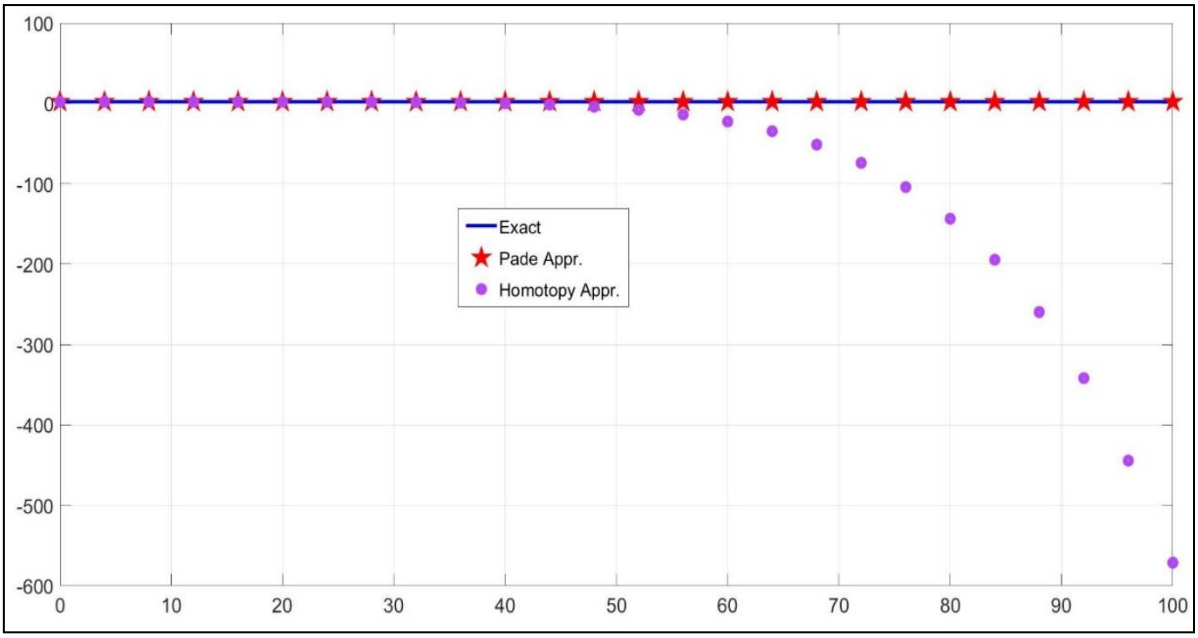

Figure 4 : Illustrate Case 4

6. Conclusion

The basic objective of this paper has been to develop an approximate solution of three - dimensional groundwater flow equation by utilizing the Pade approximation. The objective has been accomplished by utilizing the Pade approximation and comparing with homotopy perturbation method. The current work shows the extraordinary capability of the Pade approximation as illustrated in figures 1, 2, 3 and 4. Numerical results acquired utilizing the Pade approximation and the homotopy perturbation method are in agreement with exact solutions.

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