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Polynomial Approximation Method For Solving Fractional Partial Differential Equations

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ABSTRACT

This paper presents a modern approach for solving fractional partial differential equations (FPDE) which is called the polynomial approximation method based on the polynomial approximation $u_N(x,t)$ and on its general fractional derivative formula. By modifying the general fractional derivative formula of $u_N(x,t)$and with the aid of the linear FPDE, another new formula can be found for the approximation $u_N(x,t)$. This is the basic idea of the proposed method. Furthermore, the mathematical proof of the convergence and stability of this method have been studied. Some numerical examples show that the proposed method exhibits a satisfactory results.

1. Introduction

Fractional ordinary differential equation (FODE) is an equation that contains fractional derivatives of an unknown function of a single variable, while fractional partial differential equation (FPDE) is an equation that contains fractional partial derivatives of an unknown function of several variables.

Analytical solutions of FODEs and FPDEs are now a available in some special cases. But the solution

to many FDEs (ordinary and partial) will have to relay on approximate and numerical methods, just like their integer-order counterparts.

Fractional derivatives have been around for centuries but recently they have found new applications in many fields of science and engineering. Applications of fractional ordinary derivatives in viscoelasticity may be found in (1997) Diethelm [2] and in (1999) Diethelm & Freed [3] . Also, some mechanical damping models have been presented in(1998) Yuan & Agrawal [14] as FODEs, in (2001) Hanyga [4] find Multidimensional Solutions of Space - Fractional Diffusion Equations, in (2001) Schmidt & Gaul [11] Application of Fractional Calculus to Viscoelastically Damped Structures in the Finite Element Method may be found.

Moreover, fractional ordinary time derivatives have been used in (2000) Tseng, liu &Hsia [13] to compute the velocity and acceleration of some applications in signal processing, such as, radar and sonar applications.

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Applications of FPDEs are found in physics (2004) Shen & Liu [12], seismology (2002) Hanyga [5], hydrology (2005) Meerschaeh & Gaul [10], and perhaps surprisingly, FPDEs have been linked with stable distributions, where a FPDE was introduced in (2003) Lix [8] whose solution gives nearly all the stable distributions, In (2000) Meerschaert & Tadjeran [9] use Finite Difference Approximations for Two-Sided Space-Fractional Partial Differential Equations

This work is focused on solving the linear FPDEs with constant coefficients of the form:

$$D_t^{\alpha}u(x,t) + \beta\frac{\partial u(x,t)}{\partial x} = g(x,t), (x,t) \in D \tag{1}$$

when the Riemann- Liouville integral operator is invertible, $\beta \in \Re$ and D ={(x,t): c ≤ x ≤ d, a ≤ t ≤ b}.

2. New Formulation of the Approximation $u_N(x,t)$

It is popular to use the approximation

$$u_N(x,t) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_{ij}\varphi_i(x)\,\varphi_j(t) \tag{2}$$

in the two-dimensional polynomial, orthogonal polynomial, and spline approximations. Many authors and researchers used the approximation (2), such as, In (1988) Hopkins [6] which used the above approximation in two-dimensional orthogonal polynomial, and spline approximations while In (2001) Iglesias [7] used this approximation in two-dimensional spline approximation. It's popularity was due to it's simple shape, but this simple shape hide several disadvantages, such as, the difficulty in the matrix formulation of the used method (if it needed), and the number of additional terms in (2) that add worthless work.

Here a new formulation for the approximation $u_N(x,t)$will be derived. This formulation was constructed using some ideas given in (1980) Davies [1] as it will be illustrated below.

It was given in [1] that each one of the approximations of the form

$$u(x,t) \approx a_0 + a_1x + a_2t = [1\ \ x\ \ t]\{a_0\ \ a_1\ \ a_2\}$$

and

$$u(x,t) \approx a_0 + a_1x + a_2t + a_3x^2 + a_4xt + a_5t^2 + a_6x^2t + a_7xt^2 + \ldots$$
$$= [1\ \ x\ \ t\ \ x^2\ \ xt\ \ t^2\ \ x^2t\ \ xt^2\ldots]\{a_0\ \ a_1,\ldots,a_7,\ldots\}$$

has the form $u(x,t) \approx p(x,t)\,\underline{a}$

where $p(x,t)$ is a row vector of linearly independent functions, and $\underline{a}$ is a column vector of constants. For example, if we want to approximate the unknown function $u(x,t)$ in the partial differential equation:

$$\frac{\partial u(x,t)}{\partial x} + 3u(x,t) = 2x + 3(t + x^2)$$

one may guess that the following approximation could be used

$$u(x,t) \cong [1\, x\, t\, x^2]\{a_0\, a_1\, a_2\, a_3\} = a_0 + a_1x + a_2t + a_3x^2$$

Particularly, it is not easy to give definite rules which are applicable in all cases. For this reason complete polynomials are often favorite. The necessary terms for all possible polynomials up to complete quintic are shown below [1]:

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$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & x & & t & & & & \\
 & & & x^2 & & xt & & t^2 & & & \\
 & & x^3 & & x^2t & & xt^2 & & t^3 & & \\
 & x^4 & & x^3t & & x^2t^2 & & xt^3 & & t^4 & \\
x^5 & & x^4t & & x^3t^2 & & x^2t^3 & & xt^4 & & t^5 \\
 & & & & & \vdots & & & & &
\end{array}
$$

Thus a complete linear polynomial is of the form

$$a_0 + a_1x + a_2t$$

while a complete cubic polynomial is of the form

$$a_0 + a_1x + a_2t + a_3x^2 + a_4xt + a_5t^2 + a_6x^3t + a_7x^2t + a_8xt^2 + a_9t^3$$

The above ideas are the outlines that we used to establish the following new formulation $u_N(x,t)$ of the function $u(x,t)$:

$$u_N(x,t) = \sum_{j=0}^{n_1} a_j x^j + \sum_{j=1}^{n_2} a_{n_1+j} t^j + \sum_{k=1}^{n_3} \sum_{j=1}^{m_k} a_{p_k+j} x^j t^k \tag{3}$$

where $p_1 = n_1 + n_2$, $p_{k+1} = p_k + m_k$, $k = 1, 2,\ldots,n_3$, such that n_1, n_2, n_3, and m_k, for $k = 1, 2,\ldots,n_3$, are given nonnegative integers, and N is the number of terms in this approximation, i.e. N is the number of the unknowns coefficients a_j . Eq. (3) represents the general polynomial approximation that may be used to approximate $u(x,t)$. A special case is given when $n_1 = n_2 = n$, $n_3 = n - 1$, and $m_k = n - k$, as follows:

$$u_N(x,t) = \sum_{j=0}^{n} a_j x^j + \sum_{j=1}^{n} a_{n+j} t^j + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} a_{p_k+j} x^j t^k \tag{4}$$

where $p_1 = 2n$, $p_{k+1} = p_k + (n - k)$; $k = 1, 2,\ldots,n-1$, and n is a given nonnegative integer. It is obvious that eq.(4) represents the complete polynomial approximation for $u(x,t)$. To illustrate this let $n = 2$, then:

$$u_N(x,t) = \sum_{j=0}^{2} a_j x^j + \sum_{j=1}^{2} a_{j+2} t^j + \sum_{k=1}^{1} \sum_{j=1}^{2-k} a_{p_k+j} x^j t^k$$

Hence

$$u_N(x,t) = a_0 + a_1x + a_2x^2 + a_3t + a_4t^2 + a_5xt$$

which is the complete polynomial of order two.

First of all, we explore the weak form of the problem in order to build an approximation of the finite element. We multiply the first equation by an arbitrary function (test function) $v \in H_0^1(\Omega)$, integrate the result and then use the Green formula.

$$\int_\Omega f v \, dx = \int_\Omega -\Delta u \, v \, dx + \int_\Omega \boldsymbol{b} \cdot \nabla u$$

$$\int_{\Omega} fv\,dx = \int_{\Omega} \nabla u \cdot \nabla v\,dx - \underbrace{\int_{\partial\Omega} (n \cdot \nabla u)\,v\,ds}_{=0} + \int_{\Omega} \boldsymbol{b} \cdot \nabla u\,v dx$$

$$\int_{\Omega} fv\,dx = \int_{\Omega} \nabla u \cdot \nabla v\,dx + \int_{\Omega} \boldsymbol{b} \cdot \nabla u\,v dx$$

The weak formulation of $(1) - (2)$ and by inner product form is: Find $u \in H^1(\Omega)$ where

$$(\nabla u, \nabla v) + (\boldsymbol{b} \cdot \nabla u, v) = (f, v), \qquad \forall\, v \in H_0^1(\Omega) \tag{3}$$

The bilinear define $a(\cdot,\cdot) = H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

and $$a(u, v) = (\nabla u, \nabla v) + (\boldsymbol{b} \cdot \nabla u, v) \tag{4}$$

Then ,The FEM is: Find $u_h \in V_h \subset H^1(\Omega)$ such that

$$(\nabla u_h, \nabla v_h) + (\boldsymbol{b} \cdot \nabla u_h, v_h) = (f, v_h), \qquad \forall\, v_h \in V_h, \tag{5}$$

and $$a_h(u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in V_h, \tag{6}$$

where the space of finite elements

$V_h = \{v : v \text{ is continuous on } \Omega\,;\, v|_K \in P_1(K),\; K \in T_h\}$.

Assume that u is approximated over a finite element triangle K by

$$u(x, y) \approx u_h(x, y) = \sum_{j=1}^{3} u_j^K \varphi_j^K(x, y), \tag{7}$$

where u_j^K is the value of u_h at the jth node of the element, and φ_j^K is the Lagrange interpolation function, such that

$$\varphi_j^K(x_i, y_i) = \delta_{ij}.$$

We must compute the following element matrices over each element K.

Putting (7) into (5) and test function $v_h = \varphi_i^K$, $i = 1,2,3$, respectively,

and the source function f is

$$f(x, y) \approx \sum_{j=1}^{3} f_j\, \varphi_j^K(x, y), \qquad f_j = f\big(x_j, y_j\big),$$

The element diffusion matrix is obtained (stiffness matrix)

$$A_{ij} \equiv \int_K \nabla\varphi_j^K \cdot \nabla\varphi_i^K \; dx\,dy, \quad i, j = 1,2,3, \tag{8}$$

the element convection matrix

$$B_{ij} \equiv \int_K \big(\boldsymbol{b} \cdot \nabla\varphi_j^K\big)\varphi_i^K \; dx\,dy, \quad i, j = 1,2,3, \tag{9}$$

and the element mass matrix

$$M_{ij} \equiv \int_K \varphi_j^K \varphi_i^K \; dx\,dy, \quad i, j = 1,2,3, \tag{10}$$

we collect all the elements K_n, $1 \le n \le N_K$, of the grid T_h, We find a set of linear equations for the numerical solution u_j at each node:

$$\sum_{n=1}^{N_K} (A_{ij} + B_{ij}) u_j = \sum_{n=1}^{N_K} M_{ij} f_j \cdot \qquad (11)$$

For a unique of the solution, a(·,·) must be coercive provided that

$(-\frac{1}{2}\nabla \cdot \boldsymbol{b} \ge 0)$. Indeed,

$$a(v,v) = (\nabla v, \nabla v) + (\boldsymbol{b} \cdot \nabla v, v) = (|\nabla v|^2) + \left(-\frac{1}{2}\nabla \cdot \boldsymbol{b}\right) v^2 \ge C\|v\|_{1,\Omega}^2,$$

Where C is a positive constant with $|\cdot|_{1,\Omega}$ be the norm in $H^1(\Omega)$. Thus, the Lax-Milgram lemma leads to an unique special solvability.

3. New General Formula of the Fractional Derivative of $u_N(x,t)$

In this section a new general formula of the fractional derivative of the approximation was $u_N(x,t)$ established.

Proposition(3.1):

Let $\alpha \ge 0$ and $u_N(x,t)$ be the two dimensional polynomial approximation which was given in eq.(4). The fractional derivative of $u_N(x,t)$ is given by:

$$D_t^\alpha u_N(x,t) = \frac{t^{-\alpha}}{\Gamma(n-\alpha+1)} [n!\, u_N(x,t) + f(x,t)]$$

where

$$f(x,t) = \sum_{j=0}^{n} \gamma(0,n) x^j a_j + \sum_{j=1}^{n-1} j!\, \gamma(j,n) t^j a_{n+j} + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} k!\, \gamma(k,n) x^j t^k a_{p_k+j} \qquad (5)$$

and

$$\gamma(j,n) = \begin{cases} \prod_{\ell=j+1}^{n}(\ell-\alpha) - \prod_{\ell=j+1}^{n}\ell, j = 0,1,\dots,n-1 \\ 0, otherwise \end{cases} \qquad (6)$$

Proof:

Recall eq.(4):

$$u_N(x,t) = \sum_{j=0}^{n} a_j x^j + \sum_{j=1}^{n} a_{n+j} t^j + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} a_{p_k+j} x^j t^k$$

where $p_1 = 2n$, $p_{k+1} = p_k + (n-k)$; for $k = 1, 2, \dots, n-1$.

Then the 'fractional derivative' of $u_N(x,t)$ is given by

$$D_t^\alpha u_N(x,t) = \sum_{j=0}^{n} a_j \frac{x^j}{\Gamma(1-\alpha)} t^{-\alpha} + \sum_{j=1}^{n} a_{n+j} \frac{j!}{\Gamma(j-\alpha+1)} t^{j-\alpha} + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} a_{p_k+j} \frac{k!x^j}{\Gamma(k-\alpha+1)} t^{k-\alpha}$$

Rearrangement the above equation to get

$$D_t^{\alpha} u_N(x,t) = \frac{t^{-\alpha}}{\Gamma(n-\alpha+1)} \left[\sum_{j=0}^{n} a_j \frac{\Gamma(n-\alpha+1)}{\Gamma(1-\alpha)} x^j + \sum_{j=1}^{n} a_{n+j} \frac{j!\,\Gamma(n-\alpha+1)}{\Gamma(j-\alpha+1)} t^j + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} a_{p_k+j} \frac{k!\,\Gamma(n-\alpha+1)}{\Gamma(k-\alpha+1)} x^j t^k \right] \quad (7)$$

Since $\Gamma(n + 1 - \alpha) = (n - \alpha)\,\Gamma(n - \alpha) = (n - \alpha)\,(n - 1 - \alpha) \cdots (3 - \alpha)\,(2 - \alpha)\,(1 - \alpha)\,\Gamma(1 - \alpha)$

then for any integer i, $0 \le i < n$ we have:

$$\Gamma(n+1-\alpha) = \prod_{\ell=i+1}^{n} (\ell - \alpha)\,\Gamma(i+1-\alpha) \quad (8)$$

Put eq.(8) into eq.(7) , then using eq. (6) to get:

$$D_t^{\alpha} u_N(x,t) = \frac{t^{-\alpha}}{\Gamma(n-\alpha+1)} \left[\sum_{j=0}^{n} a_j \left(\prod_{\ell=1}^{n} \ell + \gamma(0,n) \right) x^j + \sum_{j=1}^{n-1} a_{n+j} j! \left(\prod_{\ell=j+1}^{n} \ell + \gamma(j,n) \right) t^j + a_n n!\, t^n \right.$$
$$\left. + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} a_{p_k+j} k! \left(\prod_{\ell=k+1}^{n} \ell + \gamma(k,n) \right) x^j t^k \right]$$

Since $j!\prod_{\ell=j+1}^{n} \ell = n!$, therefore;

$$D_t^{\alpha} u_N(x,t) = \frac{t^{-\alpha}}{\Gamma(n-\alpha+1)} \left[\sum_{j=0}^{n} a_j (n! + \gamma(0,n))\, x^j + \sum_{j=1}^{n-1} a_{n+j} (n! + j!\,\gamma(j,n)) t^j + a_n n!\, t^n \right.$$
$$\left. + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} a_{p_k+j} (n! + k!\,\gamma(k,n)) x^j t^k \right]$$

$$\Rightarrow \quad D_t^{\alpha} u_N(x,t) = \frac{t^{-\alpha}}{\Gamma(n-\alpha+1)} \left[n! \left\{ \sum_{j=0}^{n} a_j x^j + \sum_{j=1}^{n} a_{n+j} t^j + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} a_{p_k+j} x^j t^k \right\} \right.$$
$$+ \left\{ \sum_{j=0}^{n} \gamma(0,n)\, x^j a_j + \sum_{j=1}^{n-1} j!\,\gamma(j,n)\, t^j a_{n+j} \right.$$
$$\left. \left. + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} k!\,\gamma(k,n) x^j t^k a_{p_k+j} \right\} \right]$$

From equations (4) and (5) we conclude that:

$$D_t^{\alpha} u_N(x,t) = \frac{t^{-\alpha}}{\Gamma(n-\alpha+1)} [n!\, u_N(x,t) + f(x,t)]$$

4. Construction of the Polynomial Approximation Method

Our aim is to solve the linear FPDE with constant coefficients (1) when the R-L integral operator is invertible. Here, the approximated solution $u_N(x,t)$ will have the form

$$u_N(x,t) = \sum_{j=0}^{n-m} a_j t^{j+m} + \sum_{k=0}^{n-m-1} \sum_{j=1}^{n-k-m} a_{p_k+j} x^j t^{k+m} \quad (9)$$

Accordingly, the fractional derivative of $u_N(x,t)$ which have been given in proposition (1) will be:

$$D_t^{\alpha} u_N(x,t) = \frac{t^{-\alpha}}{\Gamma(n-\alpha+1)}[n!\,u_N(x,t) + f(x,t)] \tag{10}$$

where

$$f(x,t) = \sum_{j=0}^{n-m-1}(j+m)!\,\gamma(j+m,n)t^{j+m}a_j + \sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}(k+m)!\,\gamma(k+m,n)x^j t^{k+m}a_{p_k+j} \tag{11}$$

and $u_N(x,t)$ is defined in eq.(9).

Now, recall eq.(1):

$$D_t^{\alpha} u(x,t) + \beta\frac{\partial u(x,t)}{\partial x} = g(x,t) \qquad ,(x,t) \in D$$

where D = {(x,t): $0 \le x \le d$, $0 \le t \le b$}.

Differentiate eq.(9) with respect to x and put the result with eq.(10) into eq.(1) to get:

$$\frac{t^{-\alpha}}{\Gamma(n-\alpha+1)}[n!\,u_N(x,t) + f(x,t)] + \beta\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m} a_{p_k+j}\, j x^{j-1} t^{k+m} = g(x,t)$$
(12)

Simple arrangements in eq.(12) yield:

$$u_N(x,t) = G(x,t) - \frac{1}{n!}F(x,t) \tag{13}$$

where

$$\left.\begin{aligned} G(x,t) &= \frac{\Gamma(n-\alpha+1)t^{\alpha}}{n!}g(x,t) \\ F(x,t) &= f(x,t) + \beta\Gamma(n-\alpha-1)t^{\alpha}\left[\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m} a_{p_k+j}\, j x^{j-1} t^{k+m}\right] \end{aligned}\right\} \tag{14}$$

and $f(x,t)$ is defined in eq.(11).

Now, equations (9) and (13) will be used to find the unknown coefficients a_j's. Let us first consider the unknowns a_{p_k+j}, for $k = 0,\ldots n-m-1$; $j=0,\ldots n-k-m$. Since when $n = m$ such terms do not exist in the approximated solution $u_N(x,t)$, we shall find equations for the unknowns a_{p_k+j} for all $n \ge m+1$.

It is clear that differentiating both sides of equations (9) and (13) with respect to t , r-times ,and with respect to x s-times, and equating them at a certain point in D will give the unknowns a_{p_k+j}. So, differentiate both sides of eq.(13) with respect to t, we obtain

$$\frac{\partial}{\partial t}u_N(x,t) = \frac{\partial}{\partial t}G(x,t) - \frac{1}{n!}\left\{\sum_{j=0}^{n-m-1}(j+m)!\,\gamma(j+m,n)(j+m)t^{j+m-1}a_j \right.$$
$$+\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-m-1}[(k+m)!\,\gamma(k+m,n)(k+m)t^{k+m-1}x^j$$
$$\left. +\beta\Gamma(n-\alpha-1)(\alpha+k+m)jt^{\alpha+k+m-1}x^{j-1}]a_{p_k+j}\right\}$$

Repeat differentiation m-times to get:

$$\frac{\partial^m}{\partial t^m}u_N(x,t)=\frac{\partial^m}{\partial t^m}G(x,t)-\frac{1}{n!}\left\{\sum_{j=0}^{n-m-1}(j+m)!\,\gamma(j+m,n)\left(\prod_{\ell=j-m+1}^{j}(\ell+m)\right)t^j a_j\right.$$
$$+\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}(k+m)!\,\gamma(k+m,n)\left(\prod_{\ell=k-m+1}^{k}(\ell+m)\right)t^k x^j a_{p_k+j}$$
$$\left.+\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}\beta\Gamma(n-\alpha+1)\left(\prod_{\ell=0}^{m-1}(\alpha+k+m-\ell)\right)jt^{\alpha+k}x^{j-1}a_{p_k+j}\right\}$$

Hence, for $r \geq m$ we have:

$$\frac{\partial^r}{\partial t^r}u_N(x,t)=\frac{\partial^r}{\partial t^r}G(x,t)-\frac{1}{n!}\Big\{\textstyle\sum_{j=r-m}^{n-m-1}(j+m)!\,\gamma(j+m,n)\left(\prod_{\ell=j-r+1}^{j}(\ell+m)\right)t^{j+m-r}a_j$$
$$+\sum_{k=r-m}^{n-m-1}\sum_{j=1}^{n-k-m}(k+m)!\,\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)t^{k+m-r}x^j a_{p_k+j}$$
$$+\beta\Gamma(n-\alpha+1)\textstyle\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)jt^{\alpha+k+m-r}x^{j-1}a_{p_k+j}\Big\} \qquad (15)$$

Now, differentiate eq.(15) with respect to x, we get

$$\frac{\partial^{r+1}}{\partial x\partial t^r}u_N(x,t)=\frac{\partial^{r+1}}{\partial x\partial t^r}G(x,t)-\frac{1}{n!}\left\{\sum_{k=r-m}^{n-m-1}s!\,(k+m)!\,\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)t^{k+m-r}a_{p_k+1}\right.$$
$$+\sum_{k=r-m}^{n-m-2}\sum_{j=2}^{n-k-m}s!\,(k+m)!\,\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)jt^{k+m-r}x^{j-1}a_{p_k+j}$$
$$\left.+\beta\Gamma(n-\alpha+1)\sum_{k=0}^{n-m-2}\sum_{j=2}^{n-k-m}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)j(j-1)t^{\alpha+k+m-r}x^{j-2}a_{p_k+j}\right\}$$

Continue this process to get:

$$\frac{\partial^{r+s}}{\partial x^s\partial t^r}u_N(x,t)=\frac{\partial^{r+s}}{\partial x^s\partial t^r}G(x,t)-\frac{1}{n!}\left\{\sum_{k=r-m}^{n-m-s}(k+m)!\,\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)t^{k+m-r}a_{p_k+s}\right.$$
$$+\sum_{k=r-m}^{n-m-s-1}\sum_{j=s+1}^{n-k-m}(k+m)!\,\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)\left(\prod_{\ell=j-s+1}^{j}\ell\right)t^{k+m-r}x^{j-s}a_{p_k+j}$$
$$\left.+\beta\Gamma(n-\alpha+1)\textstyle\sum_{k=0}^{n-m-s-1}\sum_{j=s+1}^{n-k-m}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)\left(\prod_{\ell=j-s}^{j}\ell\right)t^{\alpha+k+m-r}x^{j-s-1}a_{p_k+j}\right\} \qquad (16)$$

In the same manner we can differentiate eq.(9) with respect to t, r-times, $r \geq m$, to get:

$$\frac{\partial^r}{\partial t^r}u_N(x,t)=\textstyle\sum_{j=r-m}^{n-m}\left(\prod_{\ell=j-r+1}^{j}(\ell+m)\right)t^{j+m-r}a_j+\sum_{k=r-m}^{n-m-1}\sum_{j=1}^{n-k-m}\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)x^j t^{k+m-r}a_{p_k+j} \qquad (17)$$

Then differentiate eq.(17) with respect to x , s-times:

$$\frac{\partial^{r+s}}{\partial x^s \partial t^r} u_N(x,t) = \sum_{k=r-m}^{n-m-s} s!\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right) t^{k+m-r} a_{p_k+s} + \sum_{k=r-m}^{n-m-s-1}\sum_{j=s+1}^{n-k-m}\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)\left(\prod_{\ell=j-s+1}^{j}\ell\right) t^{k+m-r} x^{j-s} a_{p_k+j} \tag{18}$$

Now, let $(0, \Delta t)$ be any point in D which satisfies:

$$0 < \Delta t \le T; \qquad T = min\{(2(n-m+1)R)^{m-n}, 1\} \tag{19}$$

where

$$\left.\begin{array}{l} R = max\{Q_1, |\beta\Gamma(n-\alpha+1)| Q_2\} \\ Q_1 = \max\limits_{\substack{m \le r \le n \\ r-m+1 \le k \le n-m}} \left|\frac{\prod_{\ell=k-r+1}^{k}(\ell+m)[n!+(k+m)!\gamma(k+m,n)]}{r![n!+r!\gamma(r,n)]}\right| \\ Q_2 = \max\limits_{\substack{0 \le k \le n-m-1 \\ 0 \le s \le n-r \\ m \le r \le n}} \left|\frac{(s+1)\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)}{r![n!+r!\gamma(r,n)]}\right| \end{array}\right\} \tag{20}$$

Condition (19) in sure the convergence of this method as it will be illustrated later.

Equate eq.(16) with eq. (18) at the point $(0, \Delta t)$ to get:

$$\sum_{k=r-m}^{n-m-s} s!\left[\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)(\Delta t)^{k+m-r} + \frac{1}{n!}(k+m)!\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)(\Delta t)^{k+m-r}\right] a_{p_k+s} + \frac{1}{n!}\beta\Gamma(n-\alpha+1)\sum_{k=0}^{n-m-s-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)\left(\prod_{\ell=1}^{s+1}\ell\right)(\Delta t)^{\alpha+k+m-r} a_{p_k+s+1} = \left.\frac{\partial^{r+s}}{\partial x^s \partial t^r} G(x,t)\right|_{(0,\Delta t)}$$

This implies that:

$$\left[s!\,r! + s!\frac{(r!)^2}{n!}\gamma(r,n)\right] a_{p_{r-m}+s} + \frac{s!}{n!}\sum_{k=r-m+1}^{n-m-s}\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)(\Delta t)^{k+m-r}[n! + (k+m)!\gamma(k+m,n)]\, a_{p_k+s} + \frac{(s+1)!}{n!}\beta\Gamma(n-\alpha+1)\sum_{k=0}^{n-m-s-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-r} a_{p_k+s+1} = \left.\frac{\partial^{r+s}}{\partial x^s \partial t^r} G(x,t)\right|_{(0,\Delta t)} \tag{21}$$

Dividing both sides of eq.(21) by the coefficient of $a_{p_{r-m}+s}$ yield:

$$a_{p_{r-m}+s} + \frac{1}{r![n!+r!\gamma(r,n)]}\sum_{k=r-m+1}^{n-m-s}\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)(\Delta t)^{k+m-r}[n! + (k+m)!\gamma(k+m,n)]\, a_{p_k+s} + \frac{(s+1)\beta\Gamma(n-\alpha+1)}{r![n!+r!\gamma(r,n)]}\sum_{k=0}^{n-m-s-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-r} a_{p_k+s+1} = \frac{n!}{s!\,r!\,[n!+r!\gamma(r,n)]}\left.\frac{\partial^{r+s}}{\partial x^s \partial t^r} G(x,t)\right|_{(0,\Delta t)}$$

$$for \quad r = m, m+1, \ldots, n-1, s = 1,2,\ldots,n-r \tag{22}$$

Eq. (22) represents M equations with M unknowns a_{p_k+j}, where $M=\sum_{r=m}^{n-1}(n-r)$.

Our next aim is to find $(n-m+1)$" equations for the $(n-m+1)$ unknowns'

a_j, $j = 0, 1, \ldots, n-m$. To this end, equate equations (15) and (17) at the points $(0, \Delta t)$ to get:

$$\sum_{j=r-m}^{n-m}\left(\prod_{\ell=j-r+1}^{j}(\ell+m)\right)(\Delta t)^{j+m-r}a_j+\frac{1}{n!}\sum_{j=r-m}^{n-m-1}(j+m)!\,\gamma(j+m,n)\left(\prod_{\ell=j-r+1}^{j}(\ell+m)\right)(\Delta t)^{j+m-r}a_j$$
$$+\frac{\beta\Gamma(n-\alpha+1)}{n!}\sum_{k=0}^{n-m-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-r}a_{p_k+1}=\frac{\partial^r}{\partial t^r}G(x,t)\Big|_{(0,\Delta t)}$$
$$for\quad r=m,m+1,\ldots,n \tag{23}$$

Put $r = n$ in eq.(23) to get:

$$a_{n-m}+\frac{\beta\Gamma(n-\alpha+1)}{(n!)^2}\sum_{k=0}^{n-m-1}\left(\prod_{\ell=0}^{n-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-n}a_{p_k+1}=\frac{1}{n!}\frac{\partial^n}{\partial t^n}G(x,t)\Big|_{(0,\Delta t)} \tag{24}$$

Hence, for $r \le n-1$ we get:

$$\left[r!+\frac{(r!)^2\gamma(r,n)}{n!}\right]a_{r-m}+\left(\prod_{\ell=n-m-r+1}^{n-m}(\ell+m)\right)(\Delta t)^{n-r}a_{n-m}$$
$$+\sum_{j=r-m+1}^{n-m-1}\left(\prod_{\ell=j-r+1}^{j}(\ell+m)\right)(\Delta t)^{j+m-r}\left[1+\frac{1}{n!}(j+m)!\,\gamma(j+m,n)\right]a_j$$
$$\frac{\beta\Gamma(n-\alpha+1)}{n!}\sum_{k=0}^{n-m-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-r}a_{p_k+1}=\frac{\partial^r}{\partial t^r}G(x,t)\Big|_{(0,\Delta t)}$$

This implies:

$$a_{r-m}+\frac{n!\left(\prod_{\ell=n-m-r+1}^{n-m}(\ell+m)\right)(\Delta t)^{n-r}}{r![n!+r!\gamma(r,n)]}a_{n-m}$$
$$+\sum_{j=r-m+1}^{n-m-1}\left(\prod_{\ell=j+r+1}^{j}(\ell+m)\right)(\Delta t)^{j+m-r}\left[\frac{n!+(j+m)!\,\gamma(j+m,n)}{r!\,[n!+r!\,\gamma(r,n)]}\right]a_j$$
$$+\frac{\beta\Gamma(n-\alpha+1)}{r!\,[n!+r!\,\gamma(r,n)]}\sum_{k=0}^{n-m-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-r}a_{p_k+1}$$
$$=\frac{n!}{r!\,[n!+r!\,\gamma(r,n)]}\frac{\partial^r}{\partial t^r}G(x,t)\Big|_{(0,\Delta t)}$$
$$for\quad r=m,m+1,\ldots,n-1 \tag{25}$$

Eqs.(22), (24) and (25) are the N equations needed for the evaluation of the N unknowns $a_0, a_1, \ldots, a_{N-1}$. These equations may be written in matrix form as:

$$H\,\underline{a} = B \tag{26}$$

where

$H = [h_{ij}]_{N\times N}$, $H = [b_i]_N$; $i, j = 0, 1, \ldots, N-1$, $\underline{a} = (a_0, a_1, \ldots, a_{N-1})^T$

and

$$h_{ij} = \begin{cases} 1, i = j; i,j = 0,1,\ldots,N-1 \\ \left.\begin{matrix} \frac{\left(\prod_{\ell=j-i-m+1}^{j}(\ell+m)\right)(\Delta t)^{j-i}[n!+(j+m)!\gamma(j+m,n)]}{(m+i)![n!+(m+i)!\gamma(m+i,n)]}, j = i+1,\ldots,n-m \\ \frac{\beta\Gamma(n-\alpha+1)\left(\prod_{\ell=0}^{m+i-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k-i}}{(m+i)![n!+(m+i)!\gamma(m+i,n)]}, j = p_k+1; k = 0,1,\ldots,n-m-1 \end{matrix}\right\}, \\ for \quad i = 0,\ldots,n-m \\ \left.\begin{matrix} \frac{\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)(\Delta t)^{k+m-r}[n!+(k+m)!\gamma(k+m,n)]}{r![n!+r!\gamma(r,n)]}, j = p_k+s; k = r-m+1,\ldots,n-m-s \\ \frac{(s+1)\beta\Gamma(n-\alpha+1)\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-r}}{r![n!+r!\gamma(r,n)]}, j = p_k+s+1; k = 0,\ldots,n-m-s-1 \end{matrix}\right\}, \\ for \quad i = p_{r-m}+s; r = m,\ldots,n-1; s = 1,\ldots,n-r \\ 0, otherwise \end{cases} \tag{27}$$

$$b_i = \begin{cases} \frac{n!}{(m+i)!\,[n!+(m+i)!\,\gamma(m+i,n)]}\frac{\partial^{m+i}}{\partial t^{m+i}}G(x,t)\Big|_{(0,\Delta t)}, \\ \quad for \quad i = 0,1,\ldots,n-m \\ \frac{n!}{s!\,r!\,[n!+r!\,\gamma(r,n)]}\frac{\partial^{r+s}}{\partial t^{r}\partial x^{s}}G(x,t)\Big|_{(0,\Delta t)}, \\ \quad for \quad i = p_{r-m}+s; r = m,m+1,\ldots,n-1; s = 1,2,\ldots,n-r \end{cases}$$

Finally, the approximate solution $u_N(x,t)$ in eq.(9) can be obtained by solving system (26) for the unknowns a_0, a_1, …, a_{N-1} using the Jacobi or Gauss-Seidel methods.

The next two parts concerned requirements that must be met if the approximate solution (9) is to be fairly reliable approximation to the solution of the FPDE, eq.(1). These conditions are associated with two problems, stability and convergence of the approximate solution to the solution of the FPDE

5. Numerical Examples

Example (5.1):

Consider the FPDE

$D_t^{1.1}u(x,t) - \frac{\partial u(x,t)}{\partial t} = g(x,t)$, $0 \le x \le 2$, $0 \le t \le 4$

where $g(x,t) = \frac{10}{\Gamma(1.9)}xt^{0.9} + \frac{18}{\Gamma(2.9)}t^{1.9} - 5t^2$

while the exact solution is $u(x,t) = 5\,x\,t^2 + 3\,t^3$

Let $n = 3$, then

$u_3(x,t) = a_0t^2 + a_1t^3 + a_2xt^2$, since we may take any value of Δt in the interval (0 , 0.528], so let Δt = 1/3. The results of the polynomial approximation method are obtained. These results are given by a_0 = 0, a_1 = 3 and a_2 = 5.

Example (5.2):

Consider the FPDE

$$D_x^{5/2}u(x,t) - 3\frac{\partial u(x,t)}{\partial t} = g(x,t) \qquad , 0 \le x \le 1, 0 \le t \le 1$$

where $g(x,t) = (\frac{6\sqrt{t}}{\Gamma(1.5)}xt^{0.9} - 3t)e^x$, and the exact solution is $u(x,t) = t^3 e^x$.

let n = 7. Since m = 3, then we get:

$$u_{15}(x,t) = a_0t^3 + a_1t^4 + a_2t^5 + a_3t^6 + a_4t^7 + a_5xt^3 + a_6x^2t^3 + a_7x^3t^3 + a_8x^4t^3 + a_9xt^4 + a_{10}x^2t^4 + a_{11}x^3t^4 + a_{12}xt^5 + a_{13}x^2t^5 + a_{14}xt^6$$

Let $\Delta t = 1 * 10^{-15}$, or any value in (0 , $1.6288*10^{-15}$]. The results of the polynomial approximation method with the least square error and the running time are listed in table (1):

Also, more accurate results may be obtained by increasing the number of the parameters a_j's. Depending on the least square error and running time, a comparison has been made in table (2) between the exact and approximate solutions, where the approximate solution was obtained with n =10 and $\Delta t = 5 * 10^{-35}$ ($\Delta t \in (0, 8.773*10^{-35}]$).

Table (1)

x	*T*	$u(x,t) = t^3 e^x$	**Poly. Approx**
0	0	0.00000000	0.00000000
0.1	0.1	0.00110517	0.00110517
0.2	0.2	0.00977122	0.00977119
0.3	0.3	0.03644619	0.03644558
0.4	0.4	0.09547678	0.09547085
0.5	0.5	0.20609016	0.20605452
0.6	0.6	0.39357766	0.39342210
0.7	0.7	0.69071718	0.69017411
0.8	0.8	1.13947696	1.13786809
0.9	0.9	1.79305067	1.78884654
LSE			0.00002058
Running Time			0:0:3:14

Table (2)

x	***T***	$u(x,t) = t^3 e^x$	**Poly. Approx**
0	0	0.00000000	0.00000000
0.1	0.1	0.00110517	0.00110517
0.2	0.2	0.00977122	0.00977121
0.3	0.3	0.03644619	0.03644616
0.4	0.4	0.09547678	0.09547675
0.5	0.5	0.20609016	0.20609023
0.6	0.6	0.39357766	0.39357838
0.7	0.7	0.69071718	0.69072035
0.8	0.8	1.13947696	1.13948777
0.9	0.9	1.79305067	1.79308224
LSE			0.00000000
Running Time			0:0:6:27

6. Conclusion

A new efficient method, which is called the polynomial approximation method, was introduced to find the approximate solution of FPDEs. Several examples were included for illustration. The following points have been identified:

1-This method gives the exact solution the moment the unknown function is a degree polynomial n, while for other types of functions, the accuracy of the solution depends on the degree of the used approximation.

2-A disadvantage of this method is the hand evaluation of the partial derivatives of the function $G(x\ t)$.

3-An advantage of this method is the few number of computations which is clear from its short running time.

4-The convergence condition of this method gives us a range of values from which the value of Δt may be chosen. This range depends on the given values of n and m.

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