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On Fuzzy Frechet spaces

Author Name	Abstract
Abbas Mohammed Abbas Publication data: 1 / 4 /2024 **Keywords: t**-norm, fuzzy metric space, invariant fuzzy metric, S-fuzzy linear topological spaces, fuzzy locally convex, fuzzy F-space, fuzzy frechet space.	This paper defines fuzzy frechet space and presents various results that must be proven in fuzzy frechet space. and we have definitions fuzzy locally convex and S-fuzzy topological spaces, fuzzy F-space, we need to demonstrate a few results in fuzzy F-space and fuzzy locally convex spaces.

1. Introduction:

Zadeh [6] introduced the theory of fuzzy sets in 1965. Numerous academics have expanded this idea in numerous disciplines of mathematics following Zadeh's groundbreaking work, also Hakan Efe is study fuzzy metric space and topology induced by the fuzzy metric M [5]2007.

This paper is study fuzzy frechet space and we have to prove some result in fuzzy frechet space and we have definitions fuzzy locally convex and S-fuzzy topological spaces, fuzzy F-space, we must demonstrate some outcome in fuzzy locally convex and fuzzy F-space.

2. Preliminaries

Definition (2.1): [2] Assume that $*$ is a binary operation on the set I, meaning that $* : I \times I \to I$ is a function. If the following axioms hold, then $*$ is said to be the t-norm (triangular-norm) on the set I:

(1) For every an in I, $a * 1 = a$.

(2) $*$ is commutative, meaning that for any a,b in I, $a * b = b * a$.

(3) $*$ is monotone, meaning that for every a, b, c in I, if $b, c \in I$ and $b \leq c$ then $a * b \leq a * c$

(4) $*$ is associative; that is, for any a, b, and c in I, $a * (b * c) = (a * b) * c$ is referred to as a continuous t-norm if $*$ is also continuous.

The properties of the t-norm are introduced in the following theorem:

Definition (2.2) : [2] The 3-tuple $(\mathrm{X}, \mathrm{M}, *)$ is called a fuzzy metric space if X is an arbitrary set (non-empty set), $*$ is a continuous t-norm and M is a fuzzy set in $\mathrm{X} \times \mathrm{X} \times (0, \infty)$ satisfying the following conditions : for all $x, y, z \in X$ and $s, t > 0$,

(1) $\mathrm{M}(x, y, t) > 0$,

(2) $\mathrm{M}(x, y, t) = 1$ if and only if $x = y$,

(3) $\mathrm{M}(x, y, t) = \mathrm{M}\,(y, x, t)$,

(4) $\mathrm{M}(x, y, t + s) \geq \mathrm{M}(x, z, t) * \mathrm{M}(z, y, s)$,

(5) $\mathrm{M}(x, y, .): (0, \infty) \to [0, 1]$ is continuous.

Example (2.3):[5] Let X = R. Define $a * b = a\, b$ and $M(x, y, t) = \frac{t}{t + |x - y|}$ then $(X, M, *)$ is a round fuzzy metric space.

Definition (2.4) : [7] Let $(X, M, *)$ be a fuzzy metric space.

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(a) A sequence $\{x_n\}$ in X is said to be fuzzy convergent to x in X if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that $M(x_n, x, t) > 1 - \varepsilon$, for all $n \geq n_0$ (or equivalently $\lim_{n\to\infty} M(x_n, x, t) = 1$).

(b) A sequence $\{x_n\}$ in X is said to be fuzzy Cauchy sequence if for

each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all $n, m \geq n_0$

(or equivalently $\lim_{n,m\to\infty} M(x_n, x_m, t) = 1$).

(c) A fuzzy metric space is considered complete when all fuzzy Cauchy sequences in it are fuzzy convergent.

Theorem (2.5) :[4] Let $(X, M,*)$ be a fuzzy metric space.

(1) Every fuzzy convergent sequence is fuzzy Cauchy sequence.

(2) Every sequence in X has a unique limit.

Theorem (2.6): [2] Assume that (X, M,∗) is a fuzzy metric space and that the topology created by the fuzzy metric M is represented by τ M. After that, if and only if

M(x_n, x, t) → 1 as n → ∞.then for a sequence (x_n) in X, x_n → x.

Definition (2.7):[1]Consider a fuzzy metric space (X,M,∗), where $r \in (0, 1)$, t > 0, and x ∈ X. With regard to t, the open ball with center x and radius r is denoted by the set

$$B(x, r, t) = \{y \in X: M(x, y, t) > 1 - r\}.$$

Definition(2.8): [3] A subset A of a fuzzy metric space $(X,M, *)$ is said to be open if given any point $a \in$ A, there exists $0 < r < 1$, and t > 0, such that $B[a, r, t] \subseteq A$

Remark (2.9): [5] Let (X, M, *) be a fuzzy metric space. Define τ = {A ⊂ X : for each $x \in$ X, there exist t > 0 , $r \in (0, 1)$ such that $B(x, r, t) \subset A$}.Then τ is a topology on X.

Definition (2.10): [8] Given a linear space X, let K be a subset of X. If λ x +(1-λ) y ∈ K whenever x, y ∈ K and λ∈[0,1], then K is said to be convex.

Definition (2.11): [4] Assume that the fuzzy metric space is (X, M, ∗). A closed subset A of X is defined as follows: for any sequence $\{x_n\}$ in A, it must converge to $x \Longrightarrow x \in A$.

3. Fuzzy frechet space

Definition (3.1): If M is a fuzzy metric function on X and X is a linear space over F, then M is an invariant fuzzy metric on X. if

$M(x + z, y + z, t) = M(x, y, t)$ $\forall\, x, y \in X$.

Definition (3.2): A vector space X over K is a topology generated by fuzzy metric on X such that the two mappings

$+ : X \times X \longrightarrow X \quad ; \quad (x, y) \longrightarrow x + y$

$\cdot : K \times X \longrightarrow X \quad ; \quad (k,x) \longrightarrow k\ x$

are continuous on X then X said S-fuzzy linear topology on K.

Definition(3.3): If there is a convex local base, that is, if there is a local base β at 0 in X such that every member of β is open and convex in X, then an S-fuzzy topological linear space X is said to be fuzzy locally convex.

Definition (3.4): If S-fuzzy topological linear space X induces a fuzzy complete invariant, then it is referred to as a fuzzy F-space.

Definition (3.5): A S-fuzzy topological linear space X is called fuzzy frechet space if X is fuzzy locally convex and fuzzy F-space.

Example (3.6): Let X = ℝ. Define $a * b = a\,b$ and $t > 0$

$M(x, y, t) = \frac{t}{t+|x-y|}$ $\forall x, y \in \mathbb{R}$ Then ℝ is fuzzy frechet space.

Proof: Clear ℝ is S-fuzzy topological linear space because , it is topology generated by fuzzy metric $M(x, y) \longrightarrow x+y$, $M(k, y) \longrightarrow k\,y$ are continues because they are polynomial on ℝ , let $x,y \in \mathbb{R}$

$M(x+z, y+z, t) = \frac{t}{t+|x+z-y-z|} = \frac{t}{t+|x-y|} = M(x, y, t)$, then M is an invariant fuzzy metric on ℝ , $\{x_n\}$ is fuzzy Cauchy sequence in $\mathbb{R} \Longrightarrow \lim_{n,m\to\infty} M(x_n, x_m, t) = 1 \Longrightarrow \lim_{n,m\to\infty} \frac{t}{t+|x_n-x_m|} = 1$

$\Longrightarrow \lim_{n,m\to\infty} |x_n - x_m| = 0 \Longrightarrow$ $\{x_n\}$ is Cauchy sequence in ℝ,since ℝ is complet then $\lim_{n,m\to\infty} |x_n - x| = 0$, $x \in \mathbb{R} \Longrightarrow \lim_{n,m\to\infty} |x_n - x| + t = t$

Since $t > 0 \Longrightarrow \lim_{n,m\to\infty} \frac{t}{t+|x_n-x|} = 1 \Longrightarrow \lim_{n,m\to\infty} M(x_n, x, t) = 1$, $x \in \mathbb{R}$ there fore ℝ complet fuzzy metric space

$B(0, r, t) = \{y \in \mathbb{R} : M(0, y, t) > 1 - r\} = \{y \in \mathbb{R} : \frac{t}{t+|y|} > 1 - r\} = (-a,a) \Rightarrow (-a,a) = B_a(0)$,such that $a = \frac{rt}{1-r}$ and $0 < r < 1$, $\beta = \{B(0, r, t) : 0 < r < 1\}$,$\forall A$ is open set in ℝ , $0 \in A$ there is $0 < r < 1 \ni 0 \in B(0, r, t) \subseteq A$,since $(-a,a)$ is convex $\Longrightarrow$ $B(0, r, t)$ is convex $\Longrightarrow$ local base β at 0 in ℝ such that every members of β open and convex in ℝ then ℝ is fuzzy locally convex therefore ℝ is fuzzy frechet space.

Theorem (3.7): If B is a fuzzy F-space and B is a subspace of S-fuzzy topological linear space X, then B is closed of X.

Proof: Let $\{x_n\}$ in M converge to x then by Theorem(2.5) $\{x_n\}$ is fuzzy Cauchy sequence in B since B is fuzzy F-space then M is fuzzy complete then $\lim_{n\to\infty} M(x_n, y, t) = 1$ and $y \in M$,from theorem(2.5) $\{x_n\}$ has a unique limit $x = y$ then $x \in B$ there for M is closed.

Definition (3.8): Assume that X and Y are S-fuzzy topological spaces, and $g : X \longrightarrow Y$ be a function is said to be a fuzzy open function if $g(A)$ is open in Y for all A is open in X .

Definition (3.9): [1] The fuzzy metric spaces (X, M, $*$) and (Y, N, $\star$) are to be considered. An isometry is a mapping f from X to Y such that M (x, y, t) = N (f (x), f (y), t) for every x, y ∈ X and every t > 0.

Definition (3.10): [1] If there is an isometry from X onto Y, two fuzzy metric spaces (X, M, ∗) and (Y, N, ⋆) are said to be isometric.

Theorem(3.11): Let $f : X \longrightarrow Y$ is linear isometry and fuzzy open function , X and Y are S-fuzzy topological spaces generated by the induced fuzzy metric spaces (X,M_1 , ∗) and (Y,M_2, ⋆), if X is fuzzy frechet space then Y is fuzzy frechet space .

Proof: Let X is fuzzy frechet space and f linear isometry and fuzzy open ,let $\{y_n\}$ is fuzzy Cauchy sequence in Y then for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that

$$M_2(y_n, y_m , t) > 1 - \varepsilon, \text{ for all } n, m \geq n_0$$

Some x_n , $x_m \in X \ni f(x_n)= y_n$, $f(x_m)= y_m$ .since f isometry there fore $M_2(f(x_n), f(x_m), t)=M_1(x_n , x_m , t)$,then

$M_1(x_n , x_m , t) = M_2(f(x_n), f(x_m), t) = M_2(y_n, y_m , t) > 1 - \varepsilon$, for all $n, m \geq n_0$ $\Rightarrow$ $M_1(x_n , x_m , t) > 1 - \varepsilon$, , for all $n, m \geq n_0$ there fore

$\{x_n\}$ is fuzzy Cauchy sequence in X; hence, X induces fuzzy complete invariant since X is fuzzy frechet space.

$\lim_{n\to\infty} M_1(x_n, x , t) = 1$ and $x \in X$,since f isometry there fore $M_2(f(x_n), f(x), t)=M_1(x_n , x , t)$ then

$\lim_{n\to\infty} M_2(f(x_n), f(x) , t) = 1$ and $f(x) \in Y$

Let y= f (x) ,then $\lim_{n\to\infty} M_2(y_n, \mathrm{y} , t) = 1$ and y $\in Y$

Therefore, Y is complete.

Let $x, y, z \in Y$ then for some $a, b, c \in X \ni x =f(a)$, y= $f(b), z = f(c)$

Since f linear function isometry and M_1invariant fuzzy metric on X then $M_2(x + z, y + z , t) = M_2(f(a) + f(c), f(b) + f(c) , t)= M_2(f(a + c), f(b + c) , t)= M_1(a + c, b + c , t) = M_1(a , b , t) = M_2(f(a), f(b) , t)= M_2(x, \mathrm{y} , t)$

M_2invariant fuzzy metric on Y

Since X is fuzzy frechet space then X is fuzz y locally convex, there is a local base β at 0 in X such that every members of β open and convex in X , $\beta = \{ B : B$ is open and convex $\}$ since$\forall B \in \beta$ is open set in X and f is fuzzy open function then f(B) is open set in Y ,$\forall A$ is open set in X, $0 \in A$ there is $B \in \beta \ni 0 \in B \subseteq A \Longrightarrow 0 =f(0) \in f(B) \subseteq f(A)$,since B is convex then λ x $+(1 - \lambda)$ y $\in B$ whenever x, y $\in B$ and $\lambda \in[0,1] \Longrightarrow$

$f(\lambda$ x $+(1 - \lambda)$ y $) \in f(B)$,since f linear function $\Longrightarrow \lambda f$(x) $+(1 - \lambda) f$(y) $\in f(B)$, whenever f(x),f(y) $\in f(B)$ and $\lambda \in [0,1]$,therefore $f(B)$ is convex in Y , there is a local base $\{ f(B) : f(B)$ is open and convex $\}$at 0 in Y such that every members of $\{ f(B) : f(B)$ is open and convex $\}$ open and convex in Y , therefore Y is fuzzy locally convex, then Y is fuzz y frechet space .

Definition(3.12): Let A subspace of X such that X S-fuzzy linear topology ,define A is S-fuzzy topology subspace and τ_A topology generated by fuzzy metric on A such that M_A: $A \times A \times (0, \infty) \longrightarrow [0,1]$ such that $M_A(\mathrm{x}, \mathrm{y}, \mathrm{t}) = M_X(\mathrm{x}, \mathrm{y}, \mathrm{t})$ $\forall$x,y$\in A$ and M_X is fuzzy metric on X this clear M_A fuzzy metric on A and $\tau_A = \{A \cap U: U$ is open set in $X\}$.

Theorem (3.13): If X is a fuzzy frechet space, then A is also a fuzzy frechet subspace of S-fuzzy topological linear space X.

Proof :Let A is S-topological linear subspace of fuzzy frechet space X.

Let $x, y, z \in A$ $M_A(x + z, y + z, \mathrm{t}) = M_X(x + z, y + z, \mathrm{t})= M_X(x, y, \mathrm{t}) = M_A(x, y, \mathrm{t})$

M_A invariant fuzzy metric on A ,let $\{x_n\}$ is fuzzy Cauchy sequence in A $\Longrightarrow$ $\lim_{n,m\to\infty} M_A(x_n , x_m , t) = 1 \Longrightarrow \lim_{n,m\to\infty} M_X(x_n , x_m , t) = 1$).

Also $A \subseteq X$ then $\{x_n\}$ is fuzzy Cauchy sequence in X ,since X is fuzzy frechet space then X is complete there fore $\lim_{n,m\to\infty} M_X(x_n , x , t) = 1$

$\Longrightarrow \lim_{n,m\to\infty} M_A(x_n, x, t) = 1$,since A is closed then $x \in A$ then A is complet .Since X is fuzzy frechet space then X is fuzzy locally convex, there is a local base β at 0 in X such that every members of β open and convex in X , $\beta = \{ B : B$ is open and convex $\}$,$\forall$ U is open set in X there is $B \in \beta \ni 0 \in B \subseteq U$,since A is subspace then $0 \in A \Longrightarrow$
$0 \in A \cap B \subseteq A \cap U$,since A is subspace then from[8] A is convex ,also $\Longrightarrow$ by[9] $A \cap B$ is convex and open in A there is a local base $\{ A \cap B : A \cap B$ is open and convex $\}$at 0 in A such that every members of $\{ A \cap B : A \cap B$ is open and convex $\}$ open and convex in A , therefore A is fuzzy locally convex, then A is fuzzy frechet space .

Remark (3.14): The inverse of the theorem (3.13) is not necessarily true and the example proves it

X = ℝ. Define a $*$ b = a b and $t > 0$
M(x, y, t) $= \frac{t}{t+|x-y|}$, $\forall x, y \in \mathbb{R}$
Then ℝ is fuzzy frechet space by example (3.6) but $\mathbb{Q} \subseteq \mathbb{R}$ and M(x, y, t) $= \frac{t}{t+|x-y|}$ $\forall x, y \in \mathbb{Q}$
Since ℚ is not complet then ℚ is not fuzzy frechet space .

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