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Fuzzy Weak maximal Ideals

Authors Names

[a]*Maysoun A. Hamel*
[b]*Hatam Y. Khalaf*

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ABSTRACT

In this paper we introduced and study the notions fuzzy weak maximal ideals as a generalization of weak maximal ideals , Also,we give many basic properties about this notion.

1. Introduction

"An ideal I of a rang R is called weak maximal ideal if and only if R/I is a regulear ring [1]" .

In this paper, we fuzzily the concept to weak maximal ideals. Many properties and connections with other concepts are given .

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2. preliminaries

First we give the following :

Definition 1.1 [2]: Let R be a ring , let X: R→ [0,1] is called Fuzzy ring if and only if for each $x, y \in R$,

1. $X(x+y) \geq \min\{X(x), X(y)\}$.
2. $X(x.y) \geq \min\{X(x), X(y)\}$.
3. $X(x) = X(-x)$"

Moreover the set $X_t = \{x \in R, X(x) \geq t\}$ and $X_* = \{x \in R, X(x) = X(0)\}$ are subrings of R".

"Definition 1.2 ,[3]: A F- subset K of a ring R is called a Fuzzy ideal of R if for each $x, y \in R$:

$1 - K\ (x - y) \geq min\ \{\ K\ (x), K\ (y)\}$.

$2 - K\ (xy) \geq max\ \{\ K\ (x), K\ (y)\}$"

"*Proposition* 1.3,[3]:

A F- subset K of R is a F- ideal of R if and only if $\forall t \in [0,1]$, K_t , is an ideal of R . Moreover , if K is a F- ideal of R then $K_* = \{\ x \in R, A(x) = A(0)\}$ is an ideal of R ".

"Proposition 1.4 [4]: Let A and B be two Fuzzy ideals of R. Then:

1- AB is a F- ideal of R

2- $(AB)_t = A_t\, B_t$, $\forall\ t \in [0, 1]$ " .

3-

Corollary 1.5: Let A and B be two Fuzzy ideals of R. Then: $(AB)_* = A_* B_*$.

"Proposition1.6[4]: Let X be a F-ring of R and A be a F-ideal of R such that $A \leq X$. Then, A is a F-ideal of F-ring ".

"Definition 1.7 [4]: Let X be a F- ring of R , A is a F-ideal of X such that $A \neq X$, $A(0) = X(0)$. A is called a maximal F-ideal of X iff for any $B\ is\ F$ −ideal of X, $A \leq B \leq X$, then $A_* = B_*\ or\ B = X$".

"Definition 1.8 [5]: A F- ideal K of a ring R is called pure if and only if $K \cap J = KJ$, for each F- ideal J of R ."

[a, b]University of Baghdad, College of Education for pure Sciences, Ibn-Al-Haitham, Baghdad,Iraq, E-mail:
[a]maysoon.a.h@ihcoedu.uobaghdad.edu.iq, [b]*dr.hatamyahya*@yahoo.com

"Proposition 1.9,[5]: Let A be a F- ideal of R, then A is pure if and only if A_t is a pure ideal of R, $\forall$ t $\in$ [0, 1]".

"Definition 1.10 [5]: A F- ring X is called regular if every F- ideal of X is pure" .

"Proposition 1.11,[5]: Let X be a F- ring, X is a regular F- ring if and only if X_t is regular ring X ,$\forall$ $t \in [0,1]$".

Lemma 1.12: Let H be a F- ring of R. If H is a regular F- ring, then H_* is a regular ring. The converse hold if condition (*) hold.

Proof: If H is a F- regular ring, then H_t is a regular ring by [30]. Hence, every ideal of H_t is pure. Hence, for any F- ideal A of$H$, A_* is a pure ideal in H_* . Thus, H_* is regular ring .

Conversely, if every ideal of X_* is pure, then for each F- ideal A of $X,$ A_* is a pure in H .

Now for any F- ideal B of H , B_* is a pure in H . Hence, $A_* \cap B_* = A_*B_*$, since A_* is a pure in HX_*. It follows that $A_* \cap B_* = (A \cap B)_* = (AB)_*$

So by Condition (*) , $A \cap B = AB$, and so A is pure in H . Thus, H is regular.

Lemma 1. 13:

If G is a reguler rang, let H be a F- reing over G, then H is a F-reguler ring .

Proof:

If A be a F- ideal of X , implies $A_t \leq H_t$. But G is a reguler ring, So H_t is regular ,$\forall$ $t \in [0,1]$. Therefore, A_t is pure in H_t ,$\forall$ $t \in [0,1]$.

So A is pure in H. Thus, H is a F- regular ring .

Areeg in [6] introduced the following definition:

Definition 1. 14 ,[6]: Let X be a F- ring of a ring R and A be a F- ideal in X. Define $X/A : R/A_* \to [0,1]$ such that

$$X/A\,(a + A_*) = \begin{cases} 1 & if\ a \in A_*\ , \\ \sup\{X(a+b) & if\ b \in A_*, a \notin A_* \end{cases}$$

For all $a + A_* \in$ R $/A_*$. X/A is called a quotient F- ring of $X\ by\ A$".

Proposition .1.15,[6]: If X is a F- ring of a ring R and A is a F- ideal in X , then X/A is a F-ring in R/A_* .

Proposition 1.16 [7] : First Fuzzy isomorphism Theorem

Let X and Y be F-modules and a f be a F-homomorphism between then $X/\mathrm{Kerf} \cong$ f(X).

Proposition 1.17 [7]: Second Fuzzy isomorphism Theorem"

"Let X be a F-module of an R- module M. Let A, B be a F- submodules of X with $A \leq B$.Then B/A is a F-submodule of X/A and $(X/A)/(B/A) \cong X/B$.

3. Fuzzy Weak Maximal Ideals

Definition 2.1: Let H be a F- ring and K be a F- ideal of H , then K is called a weak maximal F-ideal if H/K is a F- regular ring.

Proposition 2.2: Let H be a F- ring of a ring M and $K$is a F- ideal of H , then K is a weak maximal F-ideal if and only if K_* is a weak maximal ideal. Provided $(H/K)_* = H_*/K_*$ and Condition (*) hold of F- ideals of H/K.

Proof: Let K be a weak maximal F- ideal of H, then H/K is a regular F-ring. Hence, $(H/K)_*$ is a regular ring .
But $(H/K)_* = H_*/ K_*$. Hence X_*/K_* is regular and so K_* is a weak maximal ideal in H_* .
Conversely ; to prove K is a weak maimal F- ideal .
If K_* is a weak maimal ideal in X_* , then H_*/K_* is a regular ring .
But $(H/K)_* = H_*/K_*$. Hence, $(X/K)_*$ is regular ring .
Thus, H/K is a regular F- ring, by Lemma 1.12.
Therefore, K is a weak maximal F- ideal.

Lemma 2.3: Let H be regular F-rng and A be a F- ideal of H , then H/A is a regular F-ring .
Proof:
Let B/A be F- ideal of H/A . To prove H/A is F- regular. For any C/A F- ideal of H/A
$(B/A) \cap (C/A) \approx (B \cap C)/A)$. But B and C are pure F- ideals of H, so $B \cap C = BC$ Hence $(B \cap C)/A) = (BC)/A) = (B/A).(C/A)$
Thus, $(B/A) \cap (C/A) = (B/A).(C/A)$. Thus H/A is a F- regular ring.

Remarks and Examples 2.4:
1) Every semimaximal F-ideal is F- week maximal ideal . Provided $(H/K)_* = H_*/K_*$.
Proof:
If K is a semimaximal ideal, then $K = {}_{i=1}^{n} \cap K_i$, K_i is a maimal F- ideal of X , $\forall\, i = 1,2 \ldots, n$.
Then, $H/K = X / {}_{i=1}^{n} \cap K_i \approx$F- ideals of $({}_{i=1}^{n} \oplus X_i /K_i) \approx B$.
But $\forall\, i = 1,2 \ldots, n.$ (H_i /K_i) is simple , hence B is semisimple and so it is reguler . Thus, H/K is a regular ring and K is a weak maimal F- ideal and
The an example show the converse is not true:
Let $H: Z \to [0,1]$, defined by $X(x) = 1 , \forall x \in Z$, let $C: Z \to [0,1]$ defined by:

$$C(x) = \begin{cases} 1 & if\ x \in 6Z, \\ 3/4 & if\ x \in 2Z - 6Z, \\ 1/2 & otherwise \end{cases}$$

It is clear that C is a F- ideal of F- rng H and $H_* = Z$, $C_* = 6Z$, which is clear that C_* is a weak maximal in H_* . Also K is a weak maxmal F- ideal in H , since $H/C: Z/6Z \approx Z_6 \to [0,1]$ and Z_6 is reguler, so by Lemma 1.13 ,H/C is regular.
But C is not a semimaxiamal ideal of H since image of C=3 [[8],Theorem 2.10]
2) A F- subideal of weak maximal ideal need not to be weak maximal F-ideal, for example:
Let $H: Z_{12} \to [0,1]$, defined by: $X(x) = 1$, $\forall x \in Z_{12}$

Let $C(x) = \begin{cases} 1 & if\ x \in <\bar{2}>, \\ 0 & otherwise \end{cases}$

And $V(x) = \begin{cases} 1 & if\ x \in <\bar{4}>, \\ 0 & otherwise \end{cases}$

C and V are F- ideals in a F- ring H and note that $C_* =<\bar{2}>, V_* = <\bar{4}>$, we have $H_* /C_* = Z_2$ is a reguler ring .Since $\frac{H}{C}: Z_{12}/C_* \approx Z_2 :\to [0,1]$ and Z_2 is a reguler ring . Hence $\frac{H}{C}$ is a reguler F- ring by Lemma 3.1.3 .
So H/C is a F- regular ring . Therefore, C is a F- weak maximal ideal .
But $(H/V)_* = Z_4\ is$ not reguler ring .
Hence, H/V is not regular F-ring .
Thus, V is not is F- weak maximal ideal of H .
3) The fuzzy direct summand of weak maximal ideal need not be necessary F- weak maximal ideal . For example:
Let $H: Z_{24} \to [0,1]$, defined by: $H(x) = 1 , \forall x \in Z_{24}$.

Let $A(x) = \begin{cases} 1 & if\ x \in\ <\bar{8}>, \\ 0 & otherwise \end{cases}$

Define $B: Z \rightarrow [0,1]$ by: $B\ (x) = \begin{cases} 1 & if\ x \in <\ \bar{6}\ >, \\ 0 & otherwise \end{cases}$

$X/A \oplus B: Z_{24}/<\bar{2}> \approx Z_2 \rightarrow [0,1]$ and $Z_2$is reguler ring , so by Lemma 1.13, $X/A \oplus B$ is regular. Hence $A \oplus B$ is weak maximal .

But $A_* =<\ \bar{8}\ >$ is not a weak maximal ideal. Therefore A is not F- weak maximal ideal.

Lemma 2.5: Every semisimple F- ring X a F- regulear ring.

Proof:

If G be semisimple F- ring , then G_t is a semisimple ring, then, G_t is semisimple ring ,$\forall\ t \in [0,1]$. Hence, G_t is regular ring ,$\forall\ t \in [0,1]$. And so G is a regular F- ring by Lemma 1.13.

Corollary 2.6:Every F- ideal of semisimple F- rng is a F- weak maximal ideal.

Proof:

Let K be F- ideal in semisimple F- ring H, So H/K is semisimple F-ring. That is H/K is a F- regular ring by Lemma 2.3. So K is a F- weak maximal ideal .

Proposition 2.7:A proper F- ideal of Boolean F-rng is a F- weak maximal ideal.

Proof:

If M is a F- Boolean ring , then M is regular F- rng [9] , therefore each F- ideal A of M , $then\ the\ equotient\ M\ by\ /A$ is reguler and so that A is a weak F- maximal ideal.

Corollary 2.8 *:*

Every F- proper ideal of F- ring regulaer is a F- weak maximal ideal .

Proof:

Let H be F- regulear ring , let $C \leq H$, by Lemma 1.13, H/C is a F- regulear ing . Therefore, C is a F- weak maximal ideal.

Proposition 2.9: If D and S be F- ideals of F- ring H such that $D \leq S$. Then, S is F- weak maximal ideal in H iff B/A is a weak F- maximal ideal in H/D .

Proof:

If $D \leq S$ and S is F- weak maximal ideal in X , then H/D is a regular F- rieng. To prove S/D is F- weak maximal ideal H/D, Since $\frac{H/D}{S/D} \approx H/\quad S$ by second fuzzy isomorphism Theorem. But H/S is a regular F- ring. Then, S/D is a weak F- maximal in H/D.

Conversely, since B/A is a F-- weak maximal in X/A, implies $\frac{X/A}{B/A} \approx X/B$, so X/B a regular F- rings. Thus, B is a F- weak maximal ideal in X.

Proposition 2.10: Let A and B be two F- ideals of F- ring X such that $A \leq B$. If A is a F- weak maximal ideal in X . Then B is a F- weak maximal ideal in X .

Proof:

If A is a F- maximal ideal in X , then X/A is a F- regular ring .

Then, by Second fuzzy isomorphism theorem, we have $(X/A)/(B/A) \approx X/B$. Hence, X/B is a F- regular by Lemma 1.13. Therefore, B is a F- weak maximal ideal.

Corollary 2.11: If A and B are two F- weak maximal ideals of F- ring X , then $A + B$ is a F- weak maximal ideal.

Corollary 2.12: If $\{A_i\}_{i=1}^{n}$ is a finite family of F- weak maximal ideals of ring X, then, $\sum_{i=1}^{n} A_i$ is a F- weak maximal ideal of X.

Proposition 2.13: The intersection of two F- weak maximal ideals A and B of F –ring X is also F- weak maximal ideal such that $A(0) = B(0)$.
Proof:
$X/(A \cap B) \approx$ fuzzy subring of $(X/A \oplus X/B)$, but X/A and X/B are F- regular rings since A , B are F- weak maximal ideals , hence $(X/A \oplus X/B)$ is F- regular ring and so any F- subring of $(X/A \oplus X/B)$ is a F- regular ring . Thus, $X/(A \cap B)$ regular and so $A \cap B$ is F- weak maximal ideal.

Remark 2.14: The intersection of an infinite of collection of F- weak maximal ideals of F- ring X need not be a F- weak maximal ideal of X , as the following example show:
Let $X: Z \to [0,1]$, defined by: $X(x) = 1 , \forall x \in Z$.

$$\text{Let } A_p(x) = \begin{cases} 1 & \text{if } x \in pZ, \text{ p is a prime number ,} \\ 0 & \text{otherwise} \end{cases}$$

However,$(\cap A_p)_* = (0)$ is not weak maximal ideal of Z, since $X_*/(\cap A_p)_* \approx Z$ which is not regular. Thus, $\cap A_p$ is not F- weak maximal ideal of X.

Proposition 2.15: If A_1 and A_2 are two F- weak maximal ideals of F- ring X_1 and X_2 respectively , then $A \oplus B$ is also F- weak maximal ideal in $X_1 \oplus X_2$.
Proof: It is similar to proof of Corollary 2.11.

Corollary 2.16: If $\{A_i\}_{i=1}^{n}$ is a F- weak maximal ideals of fuzzy ring X_i , $i = 1,2,..n$, then $\sum_{i=1}^{n} A_i$ is a F- weak maximal ideal . Provided $(X/A)_* = \frac{X_*}{A_*}$.
Proposition 2.17: If G is a reguler ring , then every a F- ideal of F ring H- over G is a weak maximal ideal.
Proof:Since G is regular , then H is a F- regular ring by Lemma 1.13 .
Hence, $\forall A \leq H$, H/A is a F- regular by Lemma 2.3. Therefore, A is a F- weak maximal ideal .

Now we shall give some characterizations of F- regular ring which needed later .
Lemma 2.18: If M be a F- ring, then M is regular if and only if every principle F- ideal of X is generated by F- idempotent singleton .
Proof:
$(\rightarrow)$ Let $A = \langle x_t \rangle \leq X$. Since X is regular , then $x_t = x_t y_k x_t$, for some $y_k \in X$, let $\lambda = \min\{t, k\}$ and $y_k x_t = (yx)_\lambda$.
Put $e = yx$, then, $(e_\lambda)^2 = (y_k x_t)(y_k x_t) = y_k(x_t y_k x_t) = y_k x_t = e_\lambda$.
Hence, e_λ is an idempotent. It is easy to show that $A = \langle e_\lambda \rangle$, since $x_t = x_t e_\lambda$, implies $A = \langle x_t \rangle \leq \langle e_\lambda \rangle$. Also $e_\lambda = y_k x_t$ implies $\langle e_\lambda \rangle \leq \langle x_t \rangle = A$
$(\leftarrow)$ To prove X is regular . let $x_t \in X$. By hypothesis $\langle x_t \rangle = \langle e_\lambda \rangle$, for some F- idempotent singleton e_λ . Then, $x_t = y_k e_\lambda = (y_k e_\lambda) e_\lambda$, but $e_\lambda \in \langle x_t \rangle$, So $e_\lambda = c_l x_t$, for some $c_l \in X$ and so $x_t = x_t c_l x_t$. Thus, X is regular.

Corollary 2.19: If M be a F- ring , let $N \leq M$. Then N is a F- weak maximal ideal of M iff every F- principal ideal of M/N is generated by idempotent fuzzy singleton .
Proof:
Let n be a F- weak maximal ideal of M, then M/N is reguler. The result follows by previous Lemma 2.18.

Lemma 2.20: If M be F- ring . Then M is regular if and only if every finitely generated F- ideal of M is a F- principal ideal such that $N(0) = N(1)$.
Proof:
Let N be finitely generated F- ideal of M ideal , hence N_t is fineitely generated ideal , $\forall\ t \in [0,1]$ [10] . But M is a F- regular ring implies M_t is reguler ring . Thus, N_* is finitely generated ideal in M_* . That is N_* is a prinecipal ideal in M_* . Therefore, N is a principal F- ideal [5].

Corollary 2.21: Let X be a F- ring. Then X is regular if and only if every F- finitely generated ideal of X is generated by F- singloten idempotent
Proof:
By Lemma 2.20 every F- finitely generated ideal of X is a F-principal ideal , hence the result follows by Lemma 2.20.

Corollary 2.22: A F- ring X is regular if and only if every F- finitely generated ideal of X is a fuzzy direct summand .
Proof:
By Lemma 2.20, X is regular if and only if every F- finitely generated is a F- principal ideal .Hence by Lemma 2.20 X is regulear if and only if every fineitely generated F- ideal generated by F- iedempotent singleton .Thus, X is regular if and only if every F- finitely generated ideal of X is a direct-F- summand.
Conversely , since every finitely generated F- ideal is a fuzzy direct summand, so every F- principal ideal of X is a direct summand.

Corollary 2.23: If M be a F- rng , $N \leq M$. Then N is a F- weak maximal ideal if and only if every finitely generated F- ideal of M/N is generated by F- idempotent singleton .
Proof:
Since A is a F- weak maximal ideal , then M/N is regular. Hence, the result follows by Corollary 2.21.

Theorem 2.24: The following statements are equivalent:
1. N is a weak maximal F- ideal .
2. Every a finitely generated F- ideal of M/N is generated by F- iedempotent singleton .
3. Every a F- principal ideal of M/N is generated by F- idempotent singleton .

Proof:
(1 ⇔ 2) It follows by Corollary 2.21.
(2 ⟹ 3) It is clear .
(3 ⟹ 1) By Lemma 2.22 , M/N is F- regular .Hence A is a weak maximal F- ideal .

Theorem 2.25 *:*
If M be a F-ring, $N \leq M$, then the following are equivalent:
1- N is a weak maximal F- ideal .
2- Every a finitely generated F- ideal of M/N is a F- direct summand .
3- Every a principal F- ideal of M/N is a fuzzy direct summand .

Proof:
(1 ⇔ 2) It follows by Corollary **2**.21
(2 ⇔ 3) It is clear since every a principal F- ideal is a finitely generated
(3 ⇔ 1) It follows by Theorem 2.24.

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