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On $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$ Spaces and Weaker Forms of $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$Spaces

Authors Names	**ABSTRACT**
Reyadh D. Ali[1,a], *Ali Abdul Sahib M.Al-Butahi*[2,b] ***Publication data:*** 30/8/2024 ***Keywords***: $\mathcal{K}\mathfrak{c}-$ spaces, $\mathfrak{g}-$ *closed and* $\mathfrak{g}-$ *compact*	In this paper, the aim was to introduce new concepts of spaces, which is named $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$spaces (every $\mathfrak{g}-$ compact subsets are closed) was presented and the properties of these spaces were studied. also investigating their relationship with some other topological spaces. Also defined Weaker Forms of $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$spaces and found the relations, theorems, and results stemming from this definition. Additionally, defined Co$-\mathfrak{g}-$ compact topologies and derived related theorems and results based on this definition. Finally, we have continuing studies on the following topological space, $\mathcal{K}\mathfrak{c}-$ spaces, $\mathcal{K}(\mathfrak{gc})-$ spaces and $\mathfrak{g}\mathcal{K}(\mathfrak{gc})-$ spaces

1. Introduction:

In topology, compact spaces and KC-spaces are of great importance in mathematics and applied sciences. Understanding the properties of these spaces provides a strong foundation for developing theories and solving problems in a wide range of fields, from pure mathematical analysis to practical applications in engineering and sciences .The definition of $\mathcal{K}\mathfrak{c}-$ space (which every compact subset is closed)was presented by [1] and new concepts were introduced through the definition of the following topological spaces $\mathcal{K}(\mathfrak{gc})-$ spaces (which every compact subset is $\mathfrak{g}-$ closed), $\mathfrak{g}\mathcal{K}(\mathfrak{gc})-$ spaces (which every $\mathfrak{g}-$ compact subset is $\mathfrak{g}-$ closed) by S. K. Jassim and H. G. Ali[2]. In this research work, the aim was to introduce new concepts of spaces, which is named $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$spaces . New definitions were also introduced, which are On Weaker Forms of $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$spaces and Co$-\mathfrak{g}-$ compact topologies.

2. Preliminaries

In this section, we presented the basic concepts, definitions, theories, and results that we need for this work.

Definition 2.1[3]: A subset $\mathcal{F}$ of a space $(\mathcal{H},\mathfrak{I})$is referred to as $\mathfrak{g}-$ closed set if $\bar{\mathcal{F}} \subseteq \mathbb{N}$ $\forall \mathbb{N} \in \mathfrak{I}$ $\ni \mathcal{F} \subseteq \mathbb{N}$. If a set is $\mathfrak{g}-$ closed, then its complement is a $\mathfrak{g}-$ open.

closed set $\Rightarrow$ $\mathfrak{g}-$ closed. The converse is not true in general.

open set $\Rightarrow$ $\mathfrak{g}-$ open. In general, the opposite is not true.

Definition2.2 [3]: A space $(\mathcal{H},\mathfrak{I})$ is known as $\mathfrak{I}_{\frac{1}{2}}-$space if $\mathfrak{g}-$ closed set $\Rightarrow$ closed set.

Now we have the following diagram

$$\mathfrak{I}_1-\text{space} \Rightarrow \mathfrak{I}_{\frac{1}{2}}-\text{space} \Rightarrow \mathfrak{I}_0-\text{space}.$$

Definition 2.3 [4]: A space $(\mathcal{H},\mathfrak{I})$ *is known as* $\mathfrak{g}-$ compact space if there is a finite subcover for each $\mathfrak{g}-$open cover of $\mathcal{H}$.

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Remark 2.4 [5]: $\mathfrak{g}-$ compact space ⇒ compact. In general, the opposite is not true, as demonstrated by the example below.

Example 2.5 [5]: Let $\mathfrak{I} = \{\emptyset, \mathcal{H}, \{\hbar\}\}$ be a topology on $\mathcal{H} = \{\hbar\} \cup \{\hbar_i : i \in \mathcal{J}\}$, $\ni \mathcal{J}$ is uncountable, . so $\mathcal{H}$ is compact space nonetheless, is not $\mathfrak{g}-$ compact , because $\{\{\hbar, \hbar_i\} : i \in \mathcal{J}\}$ is $\mathfrak{g}-$ open cover of $\mathcal{H}$ but $\mathcal{H} \neq \bigcup_{i=1}^{n} \{\hbar, \hbar_i\}$.

Definition 2.6 [6]: A set $\mathcal{M}$ in $(\mathcal{H}, \mathfrak{I})$ known as $\mathcal{F}_{\sigma}-$. closed if $\mathcal{M} = \bigcup_{s \in I} \mathcal{F}_s \ni \mathcal{F}_s$ is a closed subset of $\mathcal{H}$, $\forall s \in I$.

Definition 2.7 [7]: A space $(\mathcal{H}, \mathfrak{I})$ is known as $\mathcal{P}-$ space if $\mathcal{F}_{\sigma}-$ closed set ⇒ closed .

Theorem 2.8[4], [5], [8]:

(i)Every finite subset is $\mathfrak{g}-$ compact.

(ii) Let $(\mathcal{H}, \mathfrak{I})$ be a $\mathfrak{g}-$ compact space and $\mathcal{M} \subseteq \mathcal{H}$.If $\mathcal{M}$ is $\mathfrak{g}-$ closed, then $\mathcal{M}$ is $\mathfrak{g}-$ compact.

(iii) All $\mathfrak{I}_1$ compact spaces are $\mathfrak{g}-$ compact.

(iv) Let $(\mathcal{H}, \mathfrak{I})$ be a space and $\mathcal{M}, \mathcal{W} \subseteq \mathcal{H}$.If $\mathcal{M}$ is $\mathfrak{g}-$ compact and $\mathcal{W}$ is $\mathfrak{g}-$ closed ,then $\mathcal{M} \cap \mathcal{W}$ is $\mathfrak{g}-$ compact.

3. $\mathfrak{g}$ ($\mathcal{K}\mathfrak{c}$)-space

In this section, we introduced a new definition $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$spaces and studied its relationship with other topological spaces. We also obtained relations, results, and theorems.

Definition 3.1 : A space $(\mathcal{H}, \mathfrak{I})$ is known as $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space if $\mathfrak{g}-$ compact set ⇒ closed

Theorem3.2: Every $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space is $\mathfrak{I}_1-$space .

Proof.

Let $(\mathcal{H}, \mathfrak{I})$ be an $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space , Since $\forall \hbar \in \mathcal{H}, \{\hbar\}$ is a $\mathfrak{g}-$ compact by theorem2.8 , so $\{\hbar\}$ is a closed set(because $(\mathcal{H}, \mathfrak{I})$ is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space), Hence $\mathcal{H}$ is $\mathfrak{I}_1$.

Remark 3.3:

Reverse theorem 3.2 is incorrect. The following example illustrates this.

Example3.4:

Let $\mathfrak{I} = \{\emptyset\} \cup \{\mathcal{B} : \mathcal{B} \subseteq \mathcal{H} \ and \ \mathcal{B}^C \ is \ finite \ set\}$ be a topology on $\mathcal{H} = \mathcal{R}$. So $\mathbb{Q} \subset \mathcal{R}$ is $\mathfrak{g}-$ compact and $\mathbb{Q}$ is not closed, hence $(\mathcal{H}, \mathfrak{I})$ is $\mathfrak{I}_1$ and $(\mathcal{H}, \mathfrak{I})$ is not $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space.

Theorem3.5: Every $\mathfrak{I}_2-$space is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})$.

Proof.

lf $\mathcal{F}$ is $\mathfrak{g}-$ compact subset in a $\mathfrak{I}_2-$ space $\mathcal{H}$, thus $\mathcal{F}$ is compact (Remark 2.4) , so $\mathcal{F}$ is a closed closed . Hence $\mathcal{H}$ is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})$.

Remark 3.6:

Reverse theorem 3.5 is incorrect. The following example illustrates this.

Example 3.7:
Let $\mathcal{H} = \mathcal{R}, \mathfrak{I}_{CO} = \{\emptyset\} \cup \{\mathcal{B}: \mathcal{B} \subseteq \mathcal{H}\ and\ \mathcal{B}^C\ is$ countable $set\}$ be a topology on $\mathcal{H}$, hence $\mathcal{H}$ is $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −space and not $\mathfrak{I}_2$.

Now the diagram can be written as follows:

$\mathfrak{I}_2$ − space ⇒ $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −space ⇒ $\mathfrak{I}_1$ −space ⇒ $\mathfrak{I}_{\frac{1}{2}}$ −space

Theorem 3.8 : In $\mathfrak{I}_{\frac{1}{2}}$ −space $(\mathcal{H}, \mathfrak{I})$ $\mathfrak{g}-$ compact space ⟺ compact space.

Proof.

⇒ it's clear by Remark 2.4.

⇐ Let $(\mathcal{H}, \mathfrak{I})$ be compact space and $\{\mathcal{M}_\lambda\}$ be $\mathfrak{g}$ −open covering of $\mathcal{H}$ .Since $(\mathcal{H}, \mathfrak{I})$ is a $\mathfrak{I}_{\frac{1}{2}}$ −space,then $\{\mathcal{M}_\lambda\}$ be open covering of $\mathcal{H}$.So $\{\mathcal{M}_\lambda\}$has finite sub cover. Hence$(\mathcal{H}, \mathfrak{I})$ is $\mathfrak{g}-$ compact space.

Theorem3.9: $\mathcal{K}\mathfrak{c}$ − space ⟺ $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −space .

Proof.

Let $\mathcal{F}$ be $\mathfrak{g}-$ compact subset in $\mathcal{K}\mathfrak{c}$ − space $(\mathcal{H}, \mathfrak{I})$, then $\mathcal{F}$ is compact (Remark 2.4) , so $\mathcal{F}$ is a closed .Hence $\mathcal{H}$ is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})$.

Now the reverse, let $\mathcal{F}$ be compact subset in $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ − space $(\mathcal{H}, \mathfrak{I})$, then $\mathcal{F}$ is $\mathfrak{g}-$ compact set (Theorem3.7), so $\mathcal{F}$ is a close$d$. Hence $\mathcal{H}$ is a $\mathcal{K}\mathfrak{c}$.

Theorem3.10: Every $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −space is a $\mathfrak{g}\mathcal{K}(\mathfrak{g}\mathfrak{c})$ − space .

Proof.

If $\mathcal{F}$ is $\mathfrak{g}-$ compact subset in a $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −space $\mathcal{H}$, then $\mathcal{F}$ is a $\mathfrak{g}-$ closed. Therefore $\mathcal{H}$ is a $\mathfrak{g}\mathcal{K}(\mathfrak{g}\mathfrak{c})$.

Remark 3.11:

Reverse theorem 3.10 is incorrect. The following example illustrates this.

Example 3.12:
Let $(\mathcal{H}, \mathfrak{I})$ be finite indiscrete space , hence $\mathcal{H}$ is $\mathfrak{g}$K($\mathfrak{g}$C) − space but not $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −space , because if $\mathcal{F} \subset \mathcal{H}$ is $\mathfrak{g}-$ compact .Hence $\mathcal{F}$ is not closed and is a $\mathfrak{g}-$ closed.

Remark 3.13:

The following example we show $\mathcal{K}(\mathfrak{gc})-$ space is not $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space.

Example3.14:

Let $(\mathcal{H},\mathfrak{I})$ be finite indiscrete space , hence $\mathcal{H}$ is $\mathcal{K}(\mathfrak{gc})-$ space but not $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space , because if $\mathcal{F}\subset\mathcal{H}$ is compact ($\mathfrak{g}-$ compact), Hence $\mathcal{F}$ is not closed and is a $\mathfrak{g}-$ closed .

Theorem3.15:

From compact space $\mathcal{H}$ into a $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space K, any continuous function $\hbar$ is a closed$\mathcal{d}$.

Proof. Let $(\mathcal{H},\mathfrak{I})$ be a compact space and $\mathcal{F}\subseteq\mathcal{H}$ be a closed$\mathcal{d}$,then $\mathcal{F}$ is a compact ,thus $\hbar(\mathcal{F})$ is a compact in K ,so $\hbar(\mathcal{F})$ is a $\mathfrak{g}-$ compact by theorem 3. 8 in a $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ K. Hence $\hbar(\mathcal{F})$ is a closed .Therefore $\hbar$ a closed.

Corollay3.16:

From a $\mathfrak{g}-$ compact space H into a $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space K, any continuous function $\hbar$ is a closed.

Proof. it's clear.

Theorem3.17:

Let $(\mathcal{H},\mathfrak{I})$ be any space and$(\mathrm{K},\mathfrak{I}^*)$ be a $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space . If $\hbar:\mathcal{H}\longrightarrow\mathrm{K}$ is a continuous $1-1$ function ,then $\mathcal{H}$ is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})$.

Proof.

If $\mathcal{F}\subseteq\mathcal{H}$ is a $\mathfrak{g}-$ compact . then $\mathcal{F}$ is a compact in $\mathcal{H}$,therefore $\hbar(\mathcal{F})$ is a compact ,thus $\hbar(\mathcal{F})$ is a $\mathfrak{g}-$ compact in K. Hence $\hbar(\mathcal{F})$ is a closed (K is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space), therefore $\hbar^{-1}\big(\hbar(\mathcal{F})\big)=\mathcal{F}$ a closed$\mathcal{d}$. Hence $\mathcal{H}$ is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})$.

Corollay 3.18:

Let $(\mathcal{H},\mathfrak{I})$ be any space and$(\mathrm{K},\mathfrak{I}^*)$ be a $\mathfrak{I}_2-$ space. If $\hbar:\mathcal{H}\longrightarrow\mathrm{K}$ is a continuous $1-1$ function ,then $\mathcal{H}$ is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})$.

Proof. it's clear.

Theorem3.19:

$\mathcal{g}(\mathcal{K}\mathfrak{c})-$space is a topological property.

Proof.

Let $\mathcal{H}$ be $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space and $\hbar:\mathcal{H}\longrightarrow\mathrm{K}$ be a homeomorphism. Suppose $\mathcal{F}^*\subseteq\mathrm{K}$ is a $\mathfrak{g}-$ compact , then $\mathcal{F}^*$ is a compact ,so $\hbar^{-1}(\mathcal{F}^*)\subseteq\mathcal{H}$is a compact, therefore $\hbar^{-1}(\mathcal{F}^*)$ is a closed,thus $\hbar\big(\hbar^{-1}(\mathcal{F}^*)\big)=\mathcal{F}^*\subseteq\mathrm{K}$ is a closed. Hence K is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$space.

Theorem3.20:

$\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −space is a hereditary.

Proof.

If $\mathcal{B} \subseteq \mathcal{H}$, $(\mathcal{H},\mathfrak{I})$ is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −space and $\mathcal{M} \subseteq \mathcal{B}$ is a $\mathfrak{g}-$ compact ,then $\mathcal{M}$ is a compact, so $\mathcal{M} \subseteq \mathcal{H}$is a compact ,thus $\mathcal{M} \subseteq \mathcal{H}$ is a closed(since $\mathcal{H}$ is $\mathfrak{g}(\mathcal{K}\mathfrak{c})$),therefore $\mathcal{M} \subseteq \mathcal{B}$ is a closed. Hence $\mathcal{B}$ is a $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −space.

4. On Weaker Forms of $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −spaces

In this section, we introduced new definitions, which are called On Weaker Forms of $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ −spaces and we obtained results and theorems related to these definitions

Definition 4.1: A space $(\mathcal{H},\mathfrak{I})$ is known as

(1) $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})$ if $\mathscr{g} \subseteq \mathcal{H}$ is $\mathfrak{g}-$ compact $\mathcal{F}_\sigma -$ closed **,**then $\mathscr{g}$ is closed.
(2) $\mathfrak{g}_2(\mathcal{K}\mathfrak{c})$if $\mathscr{g} \subseteq \mathcal{H}$ is $\mathfrak{g}-$ compact, then $\overline{\mathscr{g}}$ is $\mathfrak{g}-$ compact.
(3) $\mathfrak{g}_3(\mathcal{K}\mathfrak{c})$ if $\mathscr{g} \subseteq \mathcal{H}$ is $\mathfrak{g}-$ compact**,**then $\mathscr{g}$ is an $\mathcal{F}_\sigma -$ closed.
(4) an $\mathfrak{g}_4(\mathcal{K}\mathfrak{c})$if $\mathscr{g} \subseteq \mathcal{H}$ is $\mathfrak{g}-$ compact,then $\exists\mathcal{F}$ is $\mathfrak{g}-$ compact $\mathcal{F}_\sigma -$.closed $\ni$ $\mathscr{g} \subseteq \mathcal{F} \subseteq \overline{\mathscr{g}}$.

The first theorem we obtain from the definition 4.1.

Theorem4.2:

(i) Let $(\mathcal{H},\mathfrak{I})$ be an $\mathfrak{g}(\mathcal{K}\mathfrak{c})$, then $(\mathcal{H},\mathfrak{I})$ is a $\mathfrak{g}_{\mathfrak{k}}(\mathcal{K}\mathfrak{c})$. $\mathfrak{k} = 1,2,3,4$.

(ii) Let $(\mathcal{H},,\mathfrak{I})$be an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})$ and an $\mathfrak{g}_3(\mathcal{K}\mathfrak{c})$, then$(\mathcal{H},\mathfrak{I})$ is an $\mathfrak{g}(\mathcal{K}\mathfrak{c})$

(iii) Each space which is $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})$ and $\mathfrak{g}_4(\mathcal{K}\mathfrak{c})$ is an $\mathfrak{g}_2(\mathcal{K}\mathfrak{c})$.

(iv) Every $\mathfrak{g}_2(\mathcal{K}\mathfrak{c}) -$ space is an $\mathfrak{g}_4(\mathcal{K}\mathfrak{c}) -$ space and every $\mathfrak{g}_3(\mathcal{K}\mathfrak{c}) -$ space is an $\mathfrak{g}_4(\mathcal{K}\mathfrak{c}) -$ space. .

(v) Every $\mathfrak{g}_3(\mathcal{K}\mathfrak{c}) -$ space is $\mathfrak{I}_1$ −space.

(vi) Every is $\mathfrak{g}-$ compact space is an $\mathfrak{g}_2(\mathcal{K}\mathfrak{c}) -$ space, and every $\mathfrak{g}_2(\mathcal{K}\mathfrak{c}) -$ space having a dense is $\mathfrak{g}-$ compactSubset is $\mathfrak{g}-$ compact.

(vii) the property $\mathfrak{g}_3(\mathcal{K}\mathfrak{c}) -$ space is hereditary, and the properties $\mathfrak{g}_1(\mathcal{K}\mathfrak{c}), \mathfrak{g}_2(\mathcal{K}\mathfrak{c}), \mathfrak{g}_4(\mathcal{K}\mathfrak{c})$ are hereditary on $\mathcal{F}_\sigma -$. closed$d$.

(viii) Every $\mathcal{P} -$ space is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c}) -$ space .

Proof. ,

(i) Where if $\mathfrak{k} = 1$, let $(\mathcal{H},\mathfrak{I})$ be $\mathfrak{g}(\mathcal{K}\mathfrak{c})$.If $\mathcal{F} \subseteq \mathcal{H}$ is $\mathfrak{g}-$ compactt $\mathcal{F}_\sigma -$. closed $\mathcal{H}$,then $\mathcal{F}$ is closed closed . Hence $\mathcal{H}$ is a $\mathfrak{g}_1(\mathcal{K}\mathfrak{c}) -$ space.

if $\mathfrak{k} = 2$, let $\mathcal{H}$ be $\mathfrak{g}(\mathcal{K}\mathfrak{c}) -$ space and $\mathcal{F}$ be $\mathfrak{g} -$ compactsubset of $\mathcal{H}$, then $\mathcal{F}$ is closed, so $\mathcal{F} = \bar{\mathcal{F}}$ and $\bar{\mathcal{F}}$ is $\mathfrak{g} -$ compact. Hence $\mathcal{H}$ is a $\mathfrak{g}_2(\mathcal{K}\mathfrak{c}) -$ space.

if $\mathfrak{k} = 3$, let $\mathcal{H}$ be $\mathfrak{g}(\mathcal{K}\mathfrak{c})$. If $\mathcal{F} \subseteq \mathcal{H}$ is $\mathfrak{g} -$ compact, so $\mathcal{F}$ is closed, thus $\mathcal{F}$ is $\mathcal{F}_\sigma -$.closed. Hence $\mathcal{H}$ is a $\mathfrak{g}_3(\mathcal{K}\mathfrak{c}) -$ space.

If $\mathfrak{k} = 4$, let $\mathcal{H}$ be $\mathfrak{g}(\mathcal{K}\mathfrak{c}) -$ space and $\mathcal{g}$ be $\mathfrak{g} -$ compact subset of $\mathcal{H}$, then $\mathcal{g}$ is a closed, so $\mathcal{g}$ is an $\mathcal{F}_\sigma -$.closed set, now if $\mathcal{F} = \mathcal{g}$ so $\mathcal{F}$ is an $\mathfrak{g} -$ compact$\mathcal{F}_\sigma -$.closed and thus $\mathcal{g} \subseteq \mathcal{F} \subseteq \overline{\mathcal{g}}$. Hence $\mathcal{H}$is a $\mathfrak{g}_4(\mathrm{KC}) -$ space.

(ii) If $\mathcal{F}$ is an $\mathfrak{g} -$ compact subset in $\mathcal{H}$, then $\mathcal{F}$ is an $\mathcal{F}_\sigma -$.closed (since $\mathcal{H}$ is an $\mathfrak{g}_3(\mathcal{K}\mathfrak{c})$), but $\mathcal{H}$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})$,thus $\mathcal{F}$ a closed. Hence $\mathcal{H}$ is an $\mathfrak{g}(\mathcal{K}\mathfrak{c}) -$ space.

(iii) If $\ell \subseteq \mathcal{H}$is an $\mathfrak{g} -$ compact, then $\exists\, \mathcal{F}$ which is an $\mathfrak{g} -$ compact $\mathcal{F}_\sigma -$. closed $\ni \ell \subseteq \mathcal{F} \subseteq \overline{\ell}$ (since $\mathcal{H}$ is an $\mathfrak{g}_4(\mathrm{KC})$), *so* $\mathcal{F}$ is closed(since $\mathcal{H}$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})$ $-$space), then$\mathcal{F} = \overline{\ell}$, so $\overline{\ell}$ is an

$\mathfrak{g} -$ compact. Hence $\mathcal{H}$ is an $\mathfrak{g}_2(\mathrm{KC}) -$ space.

(iv) it's clear.

(v) Let $\mathcal{H}$ be an $\mathfrak{g}_3(\mathcal{K}\mathfrak{c})$ $-$space , Since $\forall \hbar \in \mathcal{H}, \mathcal{F} = \{\hbar\}$ is a $\mathfrak{g} -$ compact subset in $\mathcal{H}$, then $\mathcal{F} = \{\hbar\}$ is an $\mathcal{F}_\sigma -$. closed, thus $\mathcal{F} = \{\hbar\}$ is close$d$ $\forall \hbar \in \mathcal{H}$. Hence $\mathcal{H}$ is $\mathfrak{I}_1$ $-$space.

(vi) Let $\mathcal{H}$ be an $\mathfrak{g} -$ compact space .If $\ell \subseteq \mathcal{H}$ is *an* $\mathfrak{g} -$ compact , then $\overline{\ell}$ is closed, as is $\mathfrak{g} -$ closed,so $\overline{\ell}$ is $\mathfrak{g} -$ compact . Hence $\mathcal{H}$ is an $\mathfrak{g}_2(\mathrm{KC}) -$ space.

(vii) Let $(\mathcal{H}, \mathfrak{I})$ be $\mathfrak{g}_3(\mathcal{K}\mathfrak{c})$, $\mathcal{B}$ be a subspace of $\mathcal{H}$ .If $\mathcal{M}^* \subseteq \mathcal{B}$ is a $\mathfrak{g} -$ compact,then $\mathcal{M}^*$ is a compact,so $\mathcal{M}^*$ is a compact, as is $\mathfrak{g} -$ compact subset of $\mathcal{H}$,then $\mathcal{M}^*$is an$\mathcal{F}_\sigma -$ closed(since $\mathcal{H}$ is $\mathfrak{g}_3(\mathcal{K}\mathfrak{c}) -$ space),so $\mathcal{M}^* = \bigcup_{s\in I} \mathcal{M}_s \ni \mathcal{M}_s$ is a closed subset of $\mathcal{H}$, $\forall s \in I$, therefore

$\mathcal{M}^* = \mathcal{M}^* \cap \mathcal{H} = \bigcup_{s\in I} \mathcal{M}_s \cap \mathcal{H} = \bigcup_{s\in I}(\mathcal{M}_s \cap \mathcal{H}) \bigcup_{s\in I} \mathcal{M}^*{}_s \ni \mathcal{M}^*{}_s$ is a close$d$ subset of $\mathcal{B}$, thus$\mathcal{M}^*$ is an $\mathcal{F}_\sigma -$ closed subset of $\mathcal{B}$. Hence $\mathcal{B}$ is an $\mathfrak{g}_3(\mathcal{K}\mathfrak{c}) -$ space .

Let $\mathcal{B}$ be$\mathcal{F}\ _\sigma -$.closed a subspace of $\mathfrak{g}_1(\mathcal{K}\mathfrak{c}) -$ space $\mathcal{H}$ and $\mathcal{M}^*$ be a $\mathfrak{g} -$ compact $\mathcal{F}_\sigma -$. closed subset of $\mathcal{B}$,then $\mathcal{M}^* = \bigcup_{s\in I} \mathcal{M}^*{}_s \ni \mathcal{M}^*{}_s$ is a closed subset of $\mathcal{B}$, $\forall s \in I$,since $\mathcal{M}^*{}_s = \mathcal{M}_s \cap \mathcal{B}$ $\forall s \in I$, $\ni$ $\mathcal{M}_s$ is a closed subset of $\mathcal{H}$, $\forall s \in I$, thus:

$\mathcal{M}^* = \bigcup_{s\in I}(\mathcal{M}_s \cap \mathcal{B}\ ) = \bigcup_{s\in I}(\mathcal{M}_s) \cap \mathcal{B} = \bigcup_{s\in I}(\mathcal{M}_s) \cap \bigcup_{r\in I} \mathcal{B}_r \ni \mathcal{B}_r$ is a closed subset of $\mathcal{H}$ (since $\mathcal{B}$ is an $\mathcal{F}\ _\sigma -$. closed),so $\mathcal{M}^* = \bigcup_{s,r\in I}(\mathcal{M}_s \cup \mathcal{B}_r)$, therefore $\mathcal{M}^*$ is a $\mathfrak{g} -$ compact $\mathcal{F}_\sigma -$. close$d$ subset of $\mathcal{H}$,which is $\mathfrak{g}_1(\mathcal{K}\mathfrak{c}) -$ space $\mathcal{H}$,so $\mathcal{M}^* \subseteq \mathcal{H}$ is a closed ,then $\mathcal{M}^* \subseteq \mathcal{B}$ is a closed. Hence $\mathcal{B}$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c}) -$ space . The remainder of the proof is done in a similar manner.

(viii) Let $\mathcal{H}$ be $\mathcal{P}$ – space. If $\mathcal{F} \subseteq \mathcal{H}$ is $\mathfrak{g} -$ compact $\mathcal{F}_\sigma -$.closed,then $\mathcal{F}$ is a closed. Hence $\mathcal{H}$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c}) -$ space.

Theorem4.3:

In $\mathfrak{I}_2$ $-$space $(\mathcal{H}, \mathfrak{I})$: $\mathcal{H}$ is an $\mathfrak{g}(\mathcal{K}\mathfrak{c}) \Leftrightarrow$. $\mathcal{H}$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})$ and an $\mathfrak{g}_2(\mathcal{K}\mathfrak{c})$

Proof.

$\Rightarrow$: Obvious by Theorem 4.2(i).

$\Leftarrow$ Suppose $\mathcal{F} \subseteq \mathcal{H}$ is $\mathfrak{g}-$ compact $\ni \mathscr{f} \notin \mathcal{F}$.

Since $\mathcal{H}$ is $\mathfrak{I}_2$, $\forall \mathfrak{X} \in \mathcal{F}$ $\exists \mathcal{O}_{\mathfrak{X}} \in \mathfrak{I} \ni \mathfrak{X} \in \mathcal{O}_{\mathfrak{X}}$ with $\mathscr{f} \notin \overline{\mathcal{O}_{\mathfrak{X}}}$. Clearly$\{\mathcal{O}_{\mathfrak{X}}; \mathfrak{X} \in \mathcal{F}\}$ is $\mathfrak{g}-$open cover of $\mathcal{F}$ and so $\exists$ a finite sub cover $\ni \mathcal{F} \subseteq \bigcup_{s=1}^{n} \mathcal{O}_{\mathfrak{X}_s} \subseteq \bigcup_{s=1}^{n} \overline{\mathcal{O}_{\mathfrak{X}_s}}$.

$\forall \mathfrak{X}_s, s = 1,2,3,\ldots,n$, $\mathcal{F} \cap \overline{\mathcal{O}_{\mathfrak{X}_s}}$ is an $\mathfrak{g}-$ compact and so $\overline{\mathcal{F} \cap \overline{\mathcal{O}_{\mathfrak{X}_s}}}$ is an $\mathfrak{g}-$ compact (since $\mathcal{H}$ is an $\mathscr{g}_2(\mathcal{K}\mathfrak{c})-$ space).

Additionally, if $\mathscr{w} = \bigcup_{s=1}^{n} \overline{\mathcal{F} \cap \overline{\mathcal{O}_{\mathfrak{X}_s}}}$,then $\mathscr{w}$ is an $\mathfrak{g}-$ compact $\mathcal{F}_\sigma -$. closed, since $\mathcal{H}$ is an $\mathscr{g}_1(\mathcal{K}\mathfrak{c})-$ space, $\mathscr{w}$ is a closed $\mathfrak{g}-$ compact s $\ni \mathscr{f} \notin \mathscr{w}$.Thus $\mathscr{f} \notin \overline{\mathcal{F}}$. Therefore $\mathcal{F}$ is closed in $\mathcal{H}$.

5- **Co$-\mathfrak{g}-$ compact topologies**

In this section, we introduced a new definition, which is called Co$-\mathfrak{g}-$ compact topologies and through this definition, we presented results and theorem

Definition 5.1: Let $(\mathcal{H},\mathfrak{I})$be a topological space. The collection $\mathfrak{gc}(\mathfrak{I}) = \{\phi\} \cup \{\mathfrak{B} \in \mathfrak{I}: \mathfrak{B}^c \text{ } is \text{ } \mathfrak{g}-$ compact in $(\mathcal{H},\mathfrak{I})\}$ is a topology on $\mathcal{H}$ with $\mathfrak{gc}(\mathfrak{I}) \subseteq \mathfrak{I}$, called the Co$-\mathfrak{g}-$ compact topology of $\mathfrak{I}$ on $\mathcal{H}$. It is evident that $\mathfrak{gc}(\mathfrak{I})$ is a topology on $\mathcal{H}$ since the $\mathfrak{g}-$ compact property is hereditary on closed set, by Theorem 1.3.7

Theorem 5. In a space $(\mathcal{H},\mathfrak{I})$: $(\mathcal{H},\mathfrak{I})$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})$ if and only if $\big(\mathcal{H},\mathfrak{gc}(\mathfrak{I})\big)$ is a $\mathcal{P}-$ space

Proof.

$\Rightarrow$: Let $\mathcal{M} = \bigcup_{s\in I} \mathcal{M}_s \ni \mathcal{M}_s$ be a closed in $\big(\mathcal{H},\mathfrak{gc}(\mathfrak{I})\big)$ .If $\mathcal{M} = \mathcal{H}$ we're completed, alternatively $\mathcal{M}_s$ is an $\mathfrak{g}-$ compact and closed in$(\mathcal{H},\mathfrak{I})$,so $\mathcal{M}$ is an $\mathfrak{g}-$ compact and closed in$(\mathcal{H},\mathfrak{I})$,thus $\mathcal{M}$ is closed in $\big(\mathcal{H},\mathfrak{gc}(\mathfrak{I})\big)$. Hence $\big(\mathcal{H},\mathfrak{gc}(\mathfrak{I})\big)$ is a $\mathcal{P}-$ space.

$\Leftarrow$: Let $\mathcal{M}$ be $\mathfrak{g}-$ compact and $\mathcal{M} = \bigcup_{s\in I} \mathcal{M}_s \ni \mathcal{M}_s$ be a closed in $(\mathcal{H},\mathfrak{I})$, then $\mathcal{M}_s$ $\forall s$ is closed in $\big(\mathcal{H},\mathfrak{gc}(\mathfrak{I})\big)$,so $\mathcal{M}$ is is closed in $\big(\mathcal{H},\mathfrak{gc}(\mathfrak{I})\big)$,thus $\mathcal{M}$ is closed in$(\mathcal{H},\mathfrak{I})$. Hence $(\mathcal{H},\mathfrak{I})$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})-$ space .

Corollary5.3:

(i) If $(\mathcal{H},\mathfrak{I})$is $\mathfrak{g}-$ compact then $\mathfrak{gc}(\mathfrak{I}) = \mathfrak{I}$.

(ii) If $(\mathcal{H},\mathfrak{I})$is an $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$ space then$\big(\mathcal{H},\mathfrak{gc}(\mathfrak{I})\big)$ is a $\mathcal{P}-$ space .

Proof. Obvious.

Theorem 5.4: In $\mathfrak{g}_3(\mathcal{K}\mathfrak{c})-$ space $(\mathcal{H},\mathfrak{I})$: $(\mathcal{H},\mathfrak{I})$ is an $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ $\Leftrightarrow$.$\big(\mathcal{H},\mathfrak{gc}(\mathfrak{I})\big)$ is a $\mathcal{P}-$ space

Proof. $\Rightarrow$: Evident by Corollary 5.3(ii).

$\Leftarrow$: If $(\mathcal{H}, \mathfrak{gc}(\mathfrak{I}))$ is a $\mathcal{P}-$ space , then $(\mathcal{H}, \mathfrak{I})$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})-$ sp by Theorem 5.2. Since $(\mathcal{H}, \mathfrak{I})$ is an $\mathfrak{g}_3(\mathcal{K}\mathfrak{c})-$ space, hence $(\mathcal{H}, \mathfrak{I})$ is an $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$ space by Theorem 4.2(ii).

Theorem 5.5:

In $\mathfrak{I}_2$ $\mathfrak{g}_2(\mathcal{K}\mathfrak{c})-$ space $(\mathcal{H}, \mathfrak{I})$: $(\mathcal{H}, \mathfrak{I})$ is an $\mathfrak{g}(\mathcal{K}\mathfrak{c})$ $\Leftrightarrow$ $(\mathcal{H}, \mathfrak{gc}(\mathfrak{I}))$ is a $\mathcal{P}-$ space

Proof. $\Rightarrow$: Evident by Corollary 5.3(ii)

$\Leftarrow$: If $(\mathcal{H}, \mathfrak{gc}(\mathfrak{I}))$ is a $\mathcal{P}-$ space, then $(\mathcal{H}, \mathfrak{I})$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})-$space by Theorem 5.2.Since $(\mathcal{H}, \mathfrak{I})$ is a $\mathfrak{I}_2$ $\mathfrak{g}_2(\mathcal{K}\mathfrak{c})-$ space. Hence $(\mathcal{H}, \mathfrak{I})$ is an $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$ space by Theorem 4.3.

Corollary5.6:

If $(\mathcal{H}, \mathfrak{I})$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})-$space then $(\mathcal{H}, \mathfrak{gc}(\mathfrak{I}))$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})-$space.

Proof. Since $(\mathcal{H}, \mathfrak{I})$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})-$space, then $(\mathcal{H}, \mathfrak{gc}(\mathfrak{I}))$ is a $\mathcal{P}-$ space by Theorem5.2.Hence $(\mathcal{H}, \mathfrak{gc}(\mathfrak{I}))$ is an $\mathfrak{g}_1(\mathcal{K}\mathfrak{c})-$space by Theorem 4.2(viii).

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